Asymptotic Expansions

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The Sage Development Team

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CONTENTS

1 The Asymptotic Ring **1**

2 Asymptotic Expansion Generators **3**

3 Supplements **5**
- 3.1 Growth Groups . . . 5
- 3.2 Term Monoids . . . 5
- 3.3 Miscellaneous . . . 5

4 Asymptotic Expansions — Table of Contents **7**
- 4.1 Asymptotic Ring . . . 7
- 4.2 Common Asymptotic Expansions . . . 39
- 4.3 (Asymptotic) Growth Groups . . . 48
- 4.4 Cartesian Products of Growth Groups . . . 74
- 4.5 (Asymptotic) Term Monoids . . . 82
- 4.6 Asymptotic Expansions — Miscellaneous . . . 112
- 4.7 Asymptotics of Multivariate Generating Series . . . 120

5 Indices and Tables **157**

Python Module Index **159**

Index **161**

CHAPTER

ONE

THE ASYMPTOTIC RING

The asymptotic ring, as well as its main documentation is contained in the module

- *Asymptotic Ring*.

CHAPTER

TWO

ASYMPTOTIC EXPANSION GENERATORS

Some common asymptotic expansions can be generated in

- *Common Asymptotic Expansions*.

CHAPTER

THREE

SUPPLEMENTS

Behind the scenes of working with asymptotic expressions a couple of additional classes and tools turn up. For instance the growth of each summand is managed in growth groups, see below.

3.1 Growth Groups

The growth of a summand of an asymptotic expression is managed in

- *(Asymptotic) Growth Groups* and
- *Cartesian Products of Growth Groups*.

3.2 Term Monoids

A summand of an asymptotic expression is basically a term out of the following monoid:

- *(Asymptotic) Term Monoids*.

3.3 Miscellaneous

Various useful functions and tools are collected in

- *Asymptotic Expansions — Miscellaneous*.

CHAPTER

FOUR

ASYMPTOTIC EXPANSIONS — TABLE OF CONTENTS

4.1 Asymptotic Ring

This module provides a ring (called *AsymptoticRing*) for computations with asymptotic expansions.

4.1.1 (Informal) Definition

An asymptotic expansion is a sum such as

$$5z^3 + 4z^2 + O(z)$$

as $z \to \infty$ or

$$3x^{42}y^2 + 7x^3y^3 + O(x^2) + O(y)$$

as x and y tend to ∞. It is a truncated series (after a finite number of terms), which approximates a function.

The summands of the asymptotic expansions are partially ordered. In this module these summands are the following:

- Exact terms $c \cdot g$ with a coefficient c and an element g of a growth group (*see below*).
- O-terms $O(g)$ (see Big O notation; also called *Bachmann–Landau notation*) for a growth group element g (*again see below*).

See the Wikipedia article on asymptotic expansions for more details. Further examples of such elements can be found *here*.

Growth Groups and Elements

The elements of a *growth group* are equipped with a partial order and usually contain a variable. Examples—the order is described below these examples—are

- elements of the form z^q for some integer or rational q (growth groups with *description strings* `z^ZZ` or `z^QQ`),
- elements of the form $\log(z)^q$ for some integer or rational q (growth groups `log(z)^ZZ` or `log(z)^QQ`),
- elements of the form a^z for some rational a (growth group `QQ^z`), or
- more sophisticated constructions like products $x^r \cdot \log(x)^s \cdot a^y \cdot y^q$ (this corresponds to an element of the growth group `x^QQ * log(x)^ZZ * QQ^y * y^QQ`).

The order in all these examples is induced by the magnitude of the elements as x, y, or z (independently) tend to ∞. For elements only using the variable z this means that $g_1 \leq g_2$ if

$$\lim_{z\to\infty} \frac{g_1}{g_2} \leq 1.$$

Note: Asymptotic rings where the variable tend to some value distinct from ∞ are not yet implemented.

To find out more about

- growth groups,
- on how they are created and
- about the above used *descriptions strings*

see the top of the module *growth group*.

4.1.2 Introductory Examples

We start this series of examples by defining two asymptotic rings.

Two Rings

A Univariate Asymptotic Ring

First, we construct the following (very simple) asymptotic ring in the variable z:

```
sage: A.<z> = AsymptoticRing(growth_group='z^QQ', coefficient_ring=ZZ); A
Asymptotic Ring <z^QQ> over Integer Ring
```

A typical element of this ring is

```
sage: A.an_element()
z^(3/2) + O(z^(1/2))
```

This element consists of two summands: the exact term with coefficient 1 and growth $z^{3/2}$ and the O-term $O(z^{1/2})$. Note that the growth of $z^{3/2}$ is larger than the growth of $z^{1/2}$ as $z \to \infty$, thus this expansion cannot be simplified (which would be done automatically, see below).

Elements can be constructed via the generator z and the function `O()`, for example

```
sage: 4*z^2 + O(z)
4*z^2 + O(z)
```

A Multivariate Asymptotic Ring

Next, we construct a more sophisticated asymptotic ring in the variables x and y by

```
sage: B.<x, y> = AsymptoticRing(growth_group='x^QQ * log(x)^ZZ * (QQ_+)^y * y^QQ',␣
↪coefficient_ring=QQ); B
Asymptotic Ring <x^QQ * log(x)^ZZ * QQ^y * y^QQ> over Rational Field
```

Again, we can look at a typical (nontrivial) element:

```
sage: B.an_element()
1/8*x^(3/2)*log(x)^3*(1/8)^y*y^(3/2) + O(x^(1/2)*log(x)*(1/2)^y*y^(1/2))
```

Again, elements can be created using the generators x and y, as well as the function `O()`:

```
sage: log(x)*y/42 + O(1/2^y)
1/42*log(x)*y + O((1/2)^y)
```

Arithmetical Operations

In this section we explain how to perform various arithmetical operations with the elements of the asymptotic rings constructed above.

The Ring Operations Plus and Times

We start our calculations in the ring

```
sage: A
Asymptotic Ring <z^QQ> over Integer Ring
```

Of course, we can perform the usual ring operations $+$ and $*$:

```
sage: z^2 + 3*z*(1-z)
-2*z^2 + 3*z
sage: (3*z + 2)^3
27*z^3 + 54*z^2 + 36*z + 8
```

In addition to that, special powers—our growth group `z^QQ` allows the exponents to be out of $\mathbf{Q}$—can also be computed:

```
sage: (z^(5/2)+z^(1/7)) * z^(-1/5)
z^(23/10) + z^(-2/35)
```

The central concepts of computations with asymptotic expansions is that the O-notation can be used. For example, we have

```
sage: z^3 + z^2 + z + O(z^2)
z^3 + O(z^2)
```

where the result is simplified automatically. A more sophisticated example is

```
sage: (z+2*z^2+3*z^3+4*z^4) * (O(z)+z^2)
4*z^6 + O(z^5)
```

Division

The asymptotic expansions support division. For example, we can expand $1/(z-1)$ to a geometric series:

```
sage: 1 / (z-1)
z^(-1) + z^(-2) + z^(-3) + z^(-4) + ... + z^(-20) + O(z^(-21))
```

A default precision (parameter `default_prec` of *AsymptoticRing*) is predefined. Thus, only the first 20 summands are calculated. However, if we only want the first 5 exact terms, we cut of the rest by using

```
sage: (1 / (z-1)).truncate(5)
z^(-1) + z^(-2) + z^(-3) + z^(-4) + z^(-5) + O(z^(-6))
```

or

```
sage: 1 / (z-1) + O(z^(-6))
z^(-1) + z^(-2) + z^(-3) + z^(-4) + z^(-5) + O(z^(-6))
```

Of course, we can work with more complicated expansions as well:

```
sage: (4*z+1) / (z^3+z^2+z+O(z^0))
4*z^(-2) - 3*z^(-3) - z^(-4) + O(z^(-5))
```

Not all elements are invertible, for instance,

```
sage: 1 / O(z)
Traceback (most recent call last):
...
ZeroDivisionError: Cannot invert O(z).
```

is not invertible, since it includes 0.

Powers, Expontials and Logarithms

It works as simple as it can be; just use the usual operators `^`, `exp` and `log`. For example, we obtain the usual series expansion of the logarithm

```
sage: -log(1-1/z)
z^(-1) + 1/2*z^(-2) + 1/3*z^(-3) + ... + O(z^(-21))
```

as $z \to \infty$.

Similarly, we can apply the exponential function of an asymptotic expansion:

```
sage: exp(1/z)
1 + z^(-1) + 1/2*z^(-2) + 1/6*z^(-3) + 1/24*z^(-4) + ... + O(z^(-20))
```

Arbitrary powers work as well; for example, we have

```
sage: (1 + 1/z + O(1/z^5))^(1 + 1/z)
1 + z^(-1) + z^(-2) + 1/2*z^(-3) + 1/3*z^(-4) + O(z^(-5))
```

Multivariate Arithmetic

Now let us move on to arithmetic in the multivariate ring

```
sage: B
Asymptotic Ring <x^QQ * log(x)^ZZ * QQ^y * y^QQ> over Rational Field
```

Todo: write this part

4.1.3 More Examples

The mathematical constant e as a limit

The base of the natural logarithm e satisfies the equation

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

By using asymptotic expansions, we obtain the more precise result

```
sage: E.<n> = AsymptoticRing(growth_group='n^ZZ', coefficient_ring=SR, default_prec=5); E
Asymptotic Ring <n^ZZ> over Symbolic Ring
sage: (1 + 1/n)^n
e - 1/2*e*n^(-1) + 11/24*e*n^(-2) - 7/16*e*n^(-3) + 2447/5760*e*n^(-4) + O(n^(-5))
```

4.1.4 Selected Technical Details

Coercions and Functorial Constructions

The *AsymptoticRing* fully supports coercion. For example, the coefficient ring is automatically extended when needed:

```
sage: A
Asymptotic Ring <z^QQ> over Integer Ring
sage: (z + 1/2).parent()
Asymptotic Ring <z^QQ> over Rational Field
```

Here, the coefficient ring was extended to allow $1/2$ as a coefficient. Another example is

```
sage: C.<c> = AsymptoticRing(growth_group='c^ZZ', coefficient_ring=ZZ['e'])
sage: C.an_element()
e^3*c^3 + O(c)
sage: C.an_element() / 7
1/7*e^3*c^3 + O(c)
```

Here the result's coefficient ring is the newly found

```
sage: (C.an_element() / 7).parent()
Asymptotic Ring <c^ZZ> over
Univariate Polynomial Ring in e over Rational Field
```

Not only the coefficient ring can be extended, but the growth group as well. For example, we can add/multiply elements of the asymptotic rings `A` and `C` to get an expansion of new asymptotic ring:

```
sage: r = c*z + c/2 + O(z); r
c*z + 1/2*c + O(z)
sage: r.parent()
Asymptotic Ring <c^ZZ * z^QQ> over
Univariate Polynomial Ring in e over Rational Field
```

Data Structures

The summands of an *asymptotic expansion* are wrapped *growth group elements*. This wrapping is done by the *term monoid module*. However, inside an *asymptotic expansion* these summands (terms) are stored together with their growth-relationship, i.e., each summand knows its direct predecessors and successors. As a data structure a special poset (namely a mutable poset) is used. We can have a look at this:

```
sage: b = x^3*y + x^2*y + x*y^2 + O(x) + O(y)
sage: print(b.summands.repr_full(reverse=True))
poset(x*y^2, x^3*y, x^2*y, O(x), O(y))
+-- oo
|   +-- no successors
|   +-- predecessors:   x*y^2, x^3*y
+-- x*y^2
|   +-- successors:   oo
|   +-- predecessors:   O(x), O(y)
+-- x^3*y
|   +-- successors:   oo
|   +-- predecessors:   x^2*y
+-- x^2*y
|   +-- successors:   x^3*y
|   +-- predecessors:   O(x), O(y)
+-- O(x)
|   +-- successors:   x*y^2, x^2*y
|   +-- predecessors:   null
+-- O(y)
|   +-- successors:   x*y^2, x^2*y
|   +-- predecessors:   null
+-- null
|   +-- successors:   O(x), O(y)
|   +-- no predecessors
```

4.1.5 Various

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- Clemens Heuberger (2016)

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4.1.6 Classes and Methods

class sage.rings.asymptotic.asymptotic_ring.**AsymptoticExpansion**(*parent*, *summands*, *simplify=True*, *convert=True*)

Bases: sage.structure.element.CommutativeAlgebraElement

Class for asymptotic expansions, i.e., the elements of an *AsymptoticRing*.

INPUT:

- `parent` – the parent of the asymptotic expansion.
- `summands` – the summands as a `MutablePoset`, which represents the underlying structure.
- `simplify` – a boolean (default: `True`). It controls automatic simplification (absorption) of the asymptotic expansion.
- `convert` – a boolean (default: `True`). If set, then the `summands` are converted to the asymptotic ring (the parent of this expansion). If not, then the summands are taken as they are. In that case, the caller must ensure that the parent of the terms is set correctly.

EXAMPLES:

There are several ways to create asymptotic expansions; usually this is done by using the corresponding *asymptotic rings*:

```
sage: R_x.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=QQ); R_x
Asymptotic Ring <x^QQ> over Rational Field
sage: R_y.<y> = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=ZZ); R_y
Asymptotic Ring <y^ZZ> over Integer Ring
```

At this point, x and y are already asymptotic expansions:

```
sage: type(x)
<class 'sage.rings.asymptotic.asymptotic_ring.AsymptoticRing_with_category.element_
↪class'>
```

The usual ring operations, but allowing rational exponents (growth group `x^QQ`) can be performed:

```
sage: x^2 + 3*(x - x^(2/5))
x^2 + 3*x - 3*x^(2/5)
sage: (3*x^(1/3) + 2)^3
27*x + 54*x^(2/3) + 36*x^(1/3) + 8
```

One of the central ideas behind computing with asymptotic expansions is that the O-notation (see Wikipedia article Big_O_notation) can be used. For example, we have:

```
sage: (x+2*x^2+3*x^3+4*x^4) * (O(x)+x^2)
4*x^6 + O(x^5)
```

In particular, `O()` can be used to construct the asymptotic expansions. With the help of the *summands()*, we can also have a look at the inner structure of an asymptotic expansion:

```
sage: expr1 = x + 2*x^2 + 3*x^3 + 4*x^4; expr2 = O(x) + x^2
sage: print(expr1.summands.repr_full())
poset(x, 2*x^2, 3*x^3, 4*x^4)
+-- null
|   +-- no predecessors
```

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```
|   +-- successors:   x
+-- x
|   +-- predecessors:   null
|   +-- successors:   2*x^2
+-- 2*x^2
|   +-- predecessors:   x
|   +-- successors:   3*x^3
+-- 3*x^3
|   +-- predecessors:   2*x^2
|   +-- successors:   4*x^4
+-- 4*x^4
|   +-- predecessors:   3*x^3
|   +-- successors:   oo
+-- oo
|   +-- predecessors:   4*x^4
|   +-- no successors
sage: print(expr2.summands.repr_full())
poset(O(x), x^2)
+-- null
|   +-- no predecessors
|   +-- successors:   O(x)
+-- O(x)
|   +-- predecessors:   null
|   +-- successors:   x^2
+-- x^2
|   +-- predecessors:   O(x)
|   +-- successors:   oo
+-- oo
|   +-- predecessors:   x^2
|   +-- no successors
sage: print((expr1 * expr2).summands.repr_full())
poset(O(x^5), 4*x^6)
+-- null
|   +-- no predecessors
|   +-- successors:   O(x^5)
+-- O(x^5)
|   +-- predecessors:   null
|   +-- successors:   4*x^6
+-- 4*x^6
|   +-- predecessors:   O(x^5)
|   +-- successors:   oo
+-- oo
|   +-- predecessors:   4*x^6
|   +-- no successors
```

In addition to the monomial growth elements from above, we can also compute with logarithmic terms (simply by constructing the appropriate growth group):

```
sage: R_log = AsymptoticRing(growth_group='log(x)^QQ', coefficient_ring=QQ)
sage: lx = R_log(log(SR.var('x')))
sage: (O(lx) + lx^3)^4
log(x)^12 + O(log(x)^10)
```

See also:

(Asymptotic) Growth Groups, *(Asymptotic) Term Monoids*, `mutable_poset`.

B(*valid_from=0*)

Convert all terms in this asymptotic expansion to B-terms.

INPUT:

- `valid_from` – dictionary mapping variable names to lower bounds for the corresponding variable. The bound implied by this term is valid when all variables are at least their corresponding lower bound. If a number is passed to `valid_from`, then the lower bounds for all variables of the asymptotic expansion are set to this number

OUTPUT:

An asymptotic expansion

EXAMPLES:

```
sage: AR.<x, z> = AsymptoticRing(growth_group='x^ZZ * z^ZZ', coefficient_
↪ring=ZZ)
sage: AR.B(2*x^2, {x: 10}) # indirect doctest
B(2*x^2, x >= 10)
sage: expr = 42*x^42 + x^10 + AR.B(x^2, 20); expr # indirect doctest
42*x^42 + x^10 + B(x^2, x >= 20, z >= 20)
sage: type(AR.B(x, 10)) # indirect doctest
<class 'sage.rings.asymptotic.asymptotic_ring.AsymptoticRing_with_category.
↪element_class'>
sage: 2*z^3 + AR.B(5*z^2, {z: 20}) # indirect doctest
2*z^3 + B(5*z^2, z >= 20)
sage: (2*x).B({x: 20})
B(2*x, x >= 20)
sage: AR.B(4*x^2*z^3, valid_from=10) # indirect doctest
B(4*x^2*z^3, x >= 10, z >= 10)
sage: AR.B(42*x^2) # indirect doctest
B(42*x^2, x >= 0, z >= 0)
```

O()

Convert all terms in this asymptotic expansion to O-terms.

INPUT:

Nothing.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: AR.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: O(x)
O(x)
sage: type(O(x))
<class 'sage.rings.asymptotic.asymptotic_ring.AsymptoticRing_with_category.
↪element_class'>
sage: expr = 42*x^42 + x^10 + O(x^2); expr
42*x^42 + x^10 + O(x^2)
```

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```
sage: expr.O()
O(x^42)
sage: (2*x).O()
O(x)
```

See also:

sage.rings.power_series_ring.PowerSeriesRing(), sage.rings.laurent_series_ring.LaurentSeriesRing().

compare_with_values(*variable*, *function*, *values*, *rescaled=True*, *ring=Real Interval Field with 53 bits of precision*)

Compute the (rescaled) difference between this asymptotic expansion and the given values.

INPUT:

- `variable` – an asymptotic expansion or a string.
- `function` – a callable or symbolic expression giving the comparison values.
- `values` – a list or iterable of values where the comparison shall be carried out.
- `rescaled` – (default: `True`) determines whether the difference is divided by the error term of the asymptotic expansion.
- `ring` – (default: `RIF`) the parent into which the difference is converted.

OUTPUT:

A list of pairs containing comparison points and (rescaled) difference values.

EXAMPLES:

```
sage: assume(SR.an_element() > 0)
sage: A.<n> = AsymptoticRing('QQ^n * n^ZZ', SR)
sage: catalan = binomial(2*x, x)/(x+1)
sage: expansion = 4^n*(1/sqrt(pi)*n^(-3/2)
....:     - 9/8/sqrt(pi)*n^(-5/2)
....:     + 145/128/sqrt(pi)*n^(-7/2) + O(n^(-9/2)))
sage: expansion.compare_with_values(n, catalan, srange(5, 10))
[(5, 0.5303924444775?),
 (6, 0.5455279498787?),
 (7, 0.556880411050?),
 (8, 0.565710587724?),
 (9, 0.572775029098?)]
sage: expansion.exact_part().compare_with_values(n, catalan, [5, 10, 20])
Traceback (most recent call last):
...
NotImplementedError: exactly one error term required
sage: expansion.exact_part().compare_with_values(n, catalan, [5, 10, 20],␣
↪rescaled=False)
[(5, 0.3886263699387?), (10, 19.1842458318?), (20, 931314.63637?)]
sage: expansion.compare_with_values(n, catalan, [5, 10, 20], rescaled=False,␣
↪ring=SR)
[(5, 168/5*sqrt(5)/sqrt(pi) - 42),
 (10, 1178112/125*sqrt(10)/sqrt(pi) - 16796),
 (20, 650486218752/125*sqrt(5)/sqrt(pi) - 6564120420)]
```

Instead of a symbolic expression, a callable function can be specified as well:

```
sage: A.<n> = AsymptoticRing('n^ZZ * log(n)^ZZ', SR)
sage: def H(n):
....:     return sum(1/k for k in srange(1, n+1))
sage: H_expansion = (log(n) + euler_gamma + 1/(2*n)
....:                - 1/(12*n^2) + O(n^-4))
sage: H_expansion.compare_with_values(n, H, srange(25, 30)) # rel tol 1e-6
[(25, -0.008326995?),
 (26, -0.008327472?),
 (27, -0.008327898?),
 (28, -0.00832828?),
 (29, -0.00832862?)]
sage: forget()
```

See also:

plot_comparison()

error_part()

Return the expansion consisting of all error terms of this expansion.

INPUT:

Nothing

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: R.<x,y> = AsymptoticRing('x^QQ * log(x)^QQ * y^QQ', QQ)
sage: (x*log(x) + y^2 + O(x) + O(y)).error_part()
O(x) + O(y)
```

exact_part()

Return the expansion consisting of all exact terms of this expansion.

INPUT:

Nothing

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing('x^QQ * log(x)^QQ', QQ)
sage: (x^2 + O(x)).exact_part()
x^2
sage: (x + log(x)/2 + O(log(x)/x)).exact_part()
x + 1/2*log(x)
```

exp(*precision=None*)

Return the exponential of (i.e., the power of e to) this asymptotic expansion.

INPUT:

- `precision` – the precision used for truncating the expansion. If `None` (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

Note: The exponential function of this expansion can only be computed exactly if the respective growth element can be constructed in the underlying growth group.

ALGORITHM:

If the corresponding growth can be constructed, return the exact exponential function. Otherwise, if this term is $o(1)$, try to expand the series and truncate according to the given precision.

Todo: As soon as L-terms are implemented, this implementation has to be adapted as well in order to yield correct results.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('(e^x)^ZZ * x^ZZ * log(x)^ZZ', SR)
sage: exp(x)
e^x
sage: exp(2*x)
(e^x)^2
sage: exp(x + log(x))
e^x*x
```

```
sage: (x^(-1)).exp(precision=7)
1 + x^(-1) + 1/2*x^(-2) + 1/6*x^(-3) + ... + O(x^(-7))
```

factorial()

Return the factorial of this asymptotic expansion.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<n> = AsymptoticRing(growth_group='n^ZZ * log(n)^ZZ', coefficient_
↪ring=ZZ, default_prec=5)
sage: n.factorial()
sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(1/2)
+ 1/12*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-1/2)
+ 1/288*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-3/2)
+ O(e^(n*log(n))*(e^n)^(-1)*n^(-5/2))
sage: _.parent()
Asymptotic Ring <(e^(n*log(n)))^QQ * (e^n)^QQ * n^QQ * log(n)^QQ>
over Symbolic Constants Subring
```

Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$:

```
sage: (2*n).factorial() / n.factorial()^2 / (n+1)  # long time
1/sqrt(pi)*(e^n)^(2*log(2))*n^(-3/2)
- 9/8/sqrt(pi)*(e^n)^(2*log(2))*n^(-5/2)
```

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```
+ 145/128/sqrt(pi)*(e^n)^(2*log(2))*n^(-7/2)
+ O((e^n)^(2*log(2))*n^(-9/2))
```

Note that this method substitutes the asymptotic expansion into Stirling's formula. This substitution has to be possible which is not always guaranteed:

```
sage: S.<s> = AsymptoticRing(growth_group='s^QQ * log(s)^QQ', coefficient_
↪ring=QQ, default_prec=4)
sage: log(s).factorial()
Traceback (most recent call last):
...
TypeError: Cannot apply the substitution rules {s: log(s)} on
sqrt(2)*sqrt(pi)*e^(s*log(s))*(e^s)^(-1)*s^(1/2)
+ O(e^(s*log(s))*(e^s)^(-1)*s^(-1/2)) in
Asymptotic Ring <(e^(s*log(s)))^QQ * (e^s)^QQ * s^QQ * log(s)^QQ>
over Symbolic Constants Subring.
...
```

See also:

Stirling()

has_same_summands(*other*)

Return whether this asymptotic expansion and `other` have the same summands.

INPUT:

- `other` – an asymptotic expansion.

OUTPUT:

A boolean.

Note: While for example `O(x) == O(x)` yields `False`, these expansions *do* have the same summands and this method returns `True`.

Moreover, this method uses the coercion model in order to find a common parent for this asymptotic expansion and `other`.

EXAMPLES:

```
sage: R_ZZ.<x_ZZ> = AsymptoticRing('x^ZZ', ZZ)
sage: R_QQ.<x_QQ> = AsymptoticRing('x^ZZ', QQ)
sage: sum(x_ZZ^k for k in range(5)) == sum(x_QQ^k for k in range(5))  #␣
↪indirect doctest
True
sage: O(x_ZZ) == O(x_QQ)
False
```

invert(*precision=None*)

Return the multiplicative inverse of this element.

INPUT:

- `precision` – the precision used for truncating the expansion. If `None` (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

Warning: Due to truncation of infinite expansions, the element returned by this method might not fulfill `el * ~el == 1`.

Todo: As soon as L-terms are implemented, this implementation has to be adapted as well in order to yield correct results.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=QQ, default_
↪prec=4)
sage: ~x
x^(-1)
sage: ~(x^42)
x^(-42)
sage: ex = ~(1 + x); ex
x^(-1) - x^(-2) + x^(-3) - x^(-4) + O(x^(-5))
sage: ex * (1+x)
1 + O(x^(-4))
sage: ~(1 + O(1/x))
1 + O(x^(-1))
```

is_exact()

Return whether all terms of this expansion are exact.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('x^QQ * log(x)^QQ', QQ)
sage: (x^2 + O(x)).is_exact()
False
sage: (x^2 - x).is_exact()
True
```

is_little_o_of_one()

Return whether this expansion is of order $o(1)$.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('x^ZZ * log(x)^ZZ', QQ)
sage: (x^4 * log(x)^(-2) + x^(-4) * log(x)^2).is_little_o_of_one()
False
```

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```
sage: (x^(-1) * log(x)^1234 + x^(-2) + O(x^(-3))).is_little_o_of_one()
True
sage: (log(x) - log(x-1)).is_little_o_of_one()
True
```

```
sage: A.<x, y> = AsymptoticRing('x^QQ * y^QQ * log(y)^ZZ', QQ)
sage: (x^(-1/16) * y^32 + x^32 * y^(-1/16)).is_little_o_of_one()
False
sage: (x^(-1) * y^(-3) + x^(-3) * y^(-1)).is_little_o_of_one()
True
sage: (x^(-1) * y / log(y)).is_little_o_of_one()
False
sage: (log(y-1)/log(y) - 1).is_little_o_of_one()
True
```

See also:

limit()

limit()

Compute the limit of this asymptotic expansion.

OUTPUT:

An element of the coefficient ring.

EXAMPLES:

```
sage: A.<s> = AsymptoticRing("s^ZZ", SR, default_prec=3)
sage: (3 + 1/s + O(1/s^2)).limit()
3
sage: ((1+1/s)^s).limit()
e
sage: (1/s).limit()
0
sage: (s + 3 + 1/s + O(1/s^2)).limit()
Traceback (most recent call last):
...
ValueError: Cannot determine limit of s + 3 + s^(-1) + O(s^(-2))
sage: (O(s^0)).limit()
Traceback (most recent call last):
...
ValueError: Cannot determine limit of O(1)
```

See also:

is_little_o_of_one()

log(*base=None*, *precision=None*, *locals=None*)

The logarithm of this asymptotic expansion.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.
- `precision` – the precision used for truncating the expansion. If `None` (default value) is used, the default precision of the parent is used.

- `locals` – a dictionary which may contain the following keys and values:
  - `'log'` – value: a function. If not used, then the usual `log` is taken.

OUTPUT:

An asymptotic expansion.

Note: Computing the logarithm of an asymptotic expansion is possible if and only if there is exactly one maximal summand in the expansion.

ALGORITHM:

If the expansion has more than one summand, the asymptotic expansion for $\log(1+t)$ as t tends to 0 is used.

Todo: As soon as L-terms are implemented, this implementation has to be adapted as well in order to yield correct results.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ * log(x)^ZZ', coefficient_
↪ring=QQ)
sage: log(x)
log(x)
sage: log(x^2)
2*log(x)
sage: log(x-1)
log(x) - x^(-1) - 1/2*x^(-2) - 1/3*x^(-3) - ... + O(x^(-21))
```

The coefficient ring is automatically extended if needed:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ * log(x)^ZZ', coefficient_
↪ring=ZZ, default_prec=3)
sage: (49*x^3-1).log()
3*log(x) + 2*log(7) - 1/49*x^(-3) - 1/4802*x^(-6) ... + O(x^(-12))
sage: _.parent()
Asymptotic Ring <x^ZZ * log(x)^ZZ> over Symbolic Ring
```

If one wants to avoid this extending to the Symbolic Ring, then the following helps:

```
sage: L.<log7> = ZZ[]
sage: def mylog(z, base=None):
....:     try:
....:         if ZZ(z).is_power_of(7):
....:             return log(ZZ(z), 7) * log7
....:     except (TypeError, ValueError):
....:         pass
....:     return log(z, base)
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ * log(x)^ZZ', coefficient_
↪ring=L, default_prec=3)
sage: (49*x^3-1).log(locals={'log': mylog})
3*log(x) + 2*log7 - 1/49*x^(-3) - 1/4802*x^(-6) ... + O(x^(-12))
```

A `log`-function can also be specified to always be used with the asymptotic ring:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ * log(x)^ZZ', coefficient_
↪ring=L, default_prec=3, locals={'log': mylog})
sage: log(49*x^3-1)
3*log(x) + 2*log7 - 1/49*x^(-3) - 1/4802*x^(-6) - 1/352947*x^(-9) + O(x^(-12))
```

map_coefficients(*f*, *new_coefficient_ring=None*)

Return the asymptotic expansion obtained by applying `f` to each coefficient of this asymptotic expansion.

INPUT:

- `f` – a callable. A coefficient c will be mapped to $f(c)$.
- `new_coefficient_ring` – (default: `None`) a ring.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<n> = AsymptoticRing(growth_group='n^ZZ', coefficient_ring=ZZ)
sage: a = n^4 + 2*n^3 + 3*n^2 + O(n)
sage: a.map_coefficients(lambda c: c+1)
2*n^4 + 3*n^3 + 4*n^2 + O(n)
sage: a.map_coefficients(lambda c: c-2)
-n^4 + n^2 + O(n)
```

monomial_coefficient(*monomial*)

Return the coefficient in the base ring of the given monomial in this expansion.

INPUT:

- `monomial` – a monomial element which can be converted into the asymptotic ring of this element

OUTPUT:

An element of the coefficient ring.

EXAMPLES:

```
sage: R.<m, n> = AsymptoticRing("m^QQ*n^QQ", QQ)
sage: ae = 13 + 42/n + 2/n/m + O(n^-2)
sage: ae.monomial_coefficient(1/n)
42
sage: ae.monomial_coefficient(1/n^3)
0
sage: R.<n> = AsymptoticRing("n^QQ", ZZ)
sage: ae.monomial_coefficient(1/n)
42
sage: ae.monomial_coefficient(1)
13
```

plot_comparison(*variable*, *function*, *values*, *rescaled=True*, *ring=Real Interval Field with 53 bits of precision*, *relative_tolerance=0.025*, ***kwargs*)

Plot the (rescaled) difference between this asymptotic expansion and the given values.

INPUT:

- `variable` – an asymptotic expansion or a string.
- `function` – a callable or symbolic expression giving the comparison values.

- `values` – a list or iterable of values where the comparison shall be carried out.
- `rescaled` – (default: `True`) determines whether the difference is divided by the error term of the asymptotic expansion.
- `ring` – (default: `RIF`) the parent into which the difference is converted.
- `relative_tolerance` – (default: `0.025`). Raise error when relative error exceeds this tolerance.

Other keyword arguments are passed to `list_plot()`.

OUTPUT:

A graphics object.

Note: If rescaled (i.e. divided by the error term), the output should be bounded.

This method is mainly meant to have an easily usable plausibility check for asymptotic expansion created in some way.

EXAMPLES:

We want to check the quality of the asymptotic expansion of the harmonic numbers:

```
sage: A.<n> = AsymptoticRing('n^ZZ * log(n)^ZZ', SR)
sage: def H(n):
....:     return sum(1/k for k in srange(1, n+1))
sage: H_expansion = (log(n) + euler_gamma + 1/(2*n)
....:                - 1/(12*n^2) + O(n^-4))
sage: H_expansion.plot_comparison(n, H, srange(1, 30))
Graphics object consisting of 1 graphics primitive
```

Alternatively, the unscaled (absolute) difference can be plotted as well:

```
sage: H_expansion.plot_comparison(n, H, srange(1, 30),
....:                             rescaled=False)
Graphics object consisting of 1 graphics primitive
```

Additional keywords are passed to `list_plot()`:

```
sage: H_expansion.plot_comparison(n, H, srange(1, 30),
....:                             plotjoined=True, marker='o',
....:                             color='green')
Graphics object consisting of 1 graphics primitive
```

See also:

compare_with_values()

pow(*exponent*, *precision=None*)

Calculate the power of this asymptotic expansion to the given `exponent`.

INPUT:

- `exponent` – an element.
- `precision` – the precision used for truncating the expansion. If `None` (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: Q.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=QQ)
sage: x^(1/7)
x^(1/7)
sage: (x^(1/2) + O(x^0))^15
x^(15/2) + O(x^7)
```

```
sage: Z.<y> = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=ZZ)
sage: y^(1/7)
y^(1/7)
sage: _.parent()
Asymptotic Ring <y^QQ> over Rational Field
sage: (y^2 + O(y))^(1/2)
y + O(1)
sage: (y^2 + O(y))^(-2)
y^(-4) + O(y^(-5))
sage: (1 + 1/y + O(1/y^3))^pi
1 + pi*y^(-1) + (1/2*pi*(pi - 1))*y^(-2) + O(y^(-3))
```

```
sage: B.<z> = AsymptoticRing(growth_group='z^QQ * log(z)^QQ', coefficient_
↪ring=QQ)
sage: (z^2 + O(z))^(1/2)
z + O(1)
```

```
sage: A.<x> = AsymptoticRing('QQ^x * x^SR * log(x)^ZZ', QQ)
sage: x * 2^x
2^x*x
sage: 5^x * 2^x
10^x
sage: 2^log(x)
x^(log(2))
sage: 2^(x + 1/x)
2^x + log(2)*2^x*x^(-1) + 1/2*log(2)^2*2^x*x^(-2) + ... + O(2^x*x^(-20))
sage: _.parent()
Asymptotic Ring <QQ^x * x^SR * log(x)^QQ * Signs^x> over Symbolic Ring
```

```
sage: C.<c> = AsymptoticRing(growth_group='QQ^c * c^QQ', coefficient_ring=QQ,␣
↪default_prec=5)
sage: (3 + 1/c^2)^c
3^c + 1/3*3^c*c^(-1) + 1/18*3^c*c^(-2) - 4/81*3^c*c^(-3)
- 35/1944*3^c*c^(-4) + O(3^c*c^(-5))
sage: _.parent()
Asymptotic Ring <QQ^c * c^QQ * Signs^c> over Rational Field
sage: (2 + (1/3)^c)^c
2^c + 1/2*(2/3)^c*c + 1/8*(2/9)^c*c^2 - 1/8*(2/9)^c*c
+ 1/48*(2/27)^c*c^3 + O((2/27)^c*c^2)
sage: _.parent()
Asymptotic Ring <QQ^c * c^QQ * Signs^c> over Rational Field
```

rpow(*base*, *precision=None*, *locals=None*)
Return the power of `base` to this asymptotic expansion.

INPUT:

- `base` – an element or `'e'`.
- `precision` – the precision used for truncating the expansion. If `None` (default value) is used, the default precision of the parent is used.
- `locals` – a dictionary which may contain the following keys and values:
  - `'log'` – value: a function. If not used, then the usual `log` is taken.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('x^ZZ', QQ)
sage: (1/x).rpow('e', precision=5)
1 + x^(-1) + 1/2*x^(-2) + 1/6*x^(-3) + 1/24*x^(-4) + O(x^(-5))
```

show()
Pretty-print this asymptotic expansion.

OUTPUT:

Nothing, the representation is printed directly on the screen.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing('QQ^x * x^QQ * log(x)^QQ', SR.subring(no_
↪variables=True))
sage: (pi/2 * 5^x * x^(42/17) - sqrt(euler_gamma) * log(x)^(-7/8)).show()
1/2*pi*5^x*x^(42/17) - sqrt(euler_gamma)*log(x)^(-7/8)
```

sqrt(*precision=None*)
Return the square root of this asymptotic expansion.

INPUT:

- `precision` – the precision used for truncating the expansion. If `None` (default value) is used, the default precision of the parent is used.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: A.<s> = AsymptoticRing(growth_group='s^QQ', coefficient_ring=QQ)
sage: s.sqrt()
s^(1/2)
sage: a = (1 + 1/s).sqrt(precision=6); a
1 + 1/2*s^(-1) - 1/8*s^(-2) + 1/16*s^(-3)
- 5/128*s^(-4) + 7/256*s^(-5) + O(s^(-6))
```

See also:

pow(), *rpow()*, *exp()*.

subs(*rules=None*, *domain=None*, ***kwds*)
Substitute the given `rules` in this asymptotic expansion.

INPUT:

- `rules` – a dictionary.
- `kwds` – keyword arguments will be added to the substitution `rules`.
- `domain` – (default: `None`) a parent. The neutral elements 0 and 1 (rules for the keys `'_zero_'` and `'_one_'`, see note box below) are taken out of this domain. If `None`, then this is determined automatically.

OUTPUT:

An object.

Note: The neutral element of the asymptotic ring is replaced by the value to the key `'_zero_'`; the neutral element of the growth group is replaced by the value to the key `'_one_'`.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing(growth_group='(e^x)^QQ * x^ZZ * log(x)^ZZ',␣
↪coefficient_ring=QQ, default_prec=5)
```

```
sage: (e^x * x^2 + log(x)).subs(x=SR('s'))
s^2*e^s + log(s)
sage: _.parent()
Symbolic Ring
```

```
sage: (x^3 + x + log(x)).subs(x=x+5).truncate(5)
x^3 + 15*x^2 + 76*x + log(x) + 130 + O(x^(-1))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Rational Field
```

```
sage: (e^x * x^2 + log(x)).subs(x=2*x)
4*(e^x)^2*x^2 + log(x) + log(2)
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^QQ * log(x)^QQ> over Symbolic Ring
```

```
sage: (x^2 + log(x)).subs(x=4*x+2).truncate(5)
16*x^2 + 16*x + log(x) + 2*log(2) + 4 + 1/2*x^(-1) + O(x^(-2))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Symbolic Ring
```

```
sage: (e^x * x^2 + log(x)).subs(x=RIF(pi))
229.534211738584?
sage: _.parent()
Real Interval Field with 53 bits of precision
```

See also:

`sage.symbolic.expression.Expression.subs()`

substitute(*rules=None*, *domain=None*, ***kwds*)
Substitute the given `rules` in this asymptotic expansion.

INPUT:

- `rules` – a dictionary.
- `kwds` – keyword arguments will be added to the substitution `rules`.
- `domain` – (default: `None`) a parent. The neutral elements 0 and 1 (rules for the keys `'_zero_'` and `'_one_'`, see note box below) are taken out of this domain. If `None`, then this is determined automatically.

OUTPUT:

An object.

Note: The neutral element of the asymptotic ring is replaced by the value to the key `'_zero_'`; the neutral element of the growth group is replaced by the value to the key `'_one_'`.

EXAMPLES:

```
sage: A.<x> = AsymptoticRing(growth_group='(e^x)^QQ * x^ZZ * log(x)^ZZ',␣
↪coefficient_ring=QQ, default_prec=5)
```

```
sage: (e^x * x^2 + log(x)).subs(x=SR('s'))
s^2*e^s + log(s)
sage: _.parent()
Symbolic Ring
```

```
sage: (x^3 + x + log(x)).subs(x=x+5).truncate(5)
x^3 + 15*x^2 + 76*x + log(x) + 130 + O(x^(-1))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Rational Field
```

```
sage: (e^x * x^2 + log(x)).subs(x=2*x)
4*(e^x)^2*x^2 + log(x) + log(2)
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^QQ * log(x)^QQ> over Symbolic Ring
```

```
sage: (x^2 + log(x)).subs(x=4*x+2).truncate(5)
16*x^2 + 16*x + log(x) + 2*log(2) + 4 + 1/2*x^(-1) + O(x^(-2))
sage: _.parent()
Asymptotic Ring <(e^x)^QQ * x^ZZ * log(x)^ZZ> over Symbolic Ring
```

```
sage: (e^x * x^2 + log(x)).subs(x=RIF(pi))
229.534211738584?
sage: _.parent()
Real Interval Field with 53 bits of precision
```

See also:

`sage.symbolic.expression.Expression.subs()`

summands

The summands of this asymptotic expansion stored in the underlying data structure (a `MutablePoset`).

EXAMPLES:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: expr = 7*x^12 + x^5 + O(x^3)
sage: expr.summands
poset(O(x^3), x^5, 7*x^12)
```

See also:

sage.data_structures.mutable_poset.MutablePoset

symbolic_expression(*R=None*)

Return this asymptotic expansion as a symbolic expression.

INPUT:

- R – (a subring of) the symbolic ring or None. The output will be an element of R. If None, then the symbolic ring is used.

OUTPUT:

A symbolic expression.

EXAMPLES:

```
sage: A.<x, y, z> = AsymptoticRing(growth_group='x^ZZ * y^QQ * log(y)^QQ * (QQ_
↪+)^z * z^QQ', coefficient_ring=QQ)
sage: SR(A.an_element())  # indirect doctest
1/8*(1/8)^z*x^3*y^(3/2)*z^(3/2)*log(y)^(3/2) +
Order((1/2)^z*x*sqrt(y)*sqrt(z)*sqrt(log(y)))

sage: A.<x, y, z> = AsymptoticRing(growth_group='x^ZZ * y^QQ * log(y)^QQ * (QQ_
↪+)^z * z^QQ', coefficient_ring=QQ)
sage: SR(A.an_element())  # indirect doctest
1/8*(1/8)^z*x^3*y^(3/2)*z^(3/2)*log(y)^(3/2) +
Order((1/2)^z*x*sqrt(y)*sqrt(z)*sqrt(log(y)))
```

truncate(*precision=None*)

Truncate this asymptotic expansion.

INPUT:

- precision – a positive integer or None. Number of summands that are kept. If None (default value) is given, then default_prec from the parent is used.

OUTPUT:

An asymptotic expansion.

Note: For example, truncating an asymptotic expansion with precision=20 does not yield an expansion with exactly 20 summands! Rather than that, it keeps the 20 summands with the largest growth, and adds appropriate O-Terms.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing('x^ZZ', QQ)
sage: ex = sum(x^k for k in range(5)); ex
x^4 + x^3 + x^2 + x + 1
sage: ex.truncate(precision=2)
```

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```
x^4 + x^3 + O(x^2)
sage: ex.truncate(precision=0)
O(x^4)
sage: ex.truncate()
x^4 + x^3 + x^2 + x + 1
```

variable_names()

Return the names of the variables of this asymptotic expansion.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: A.<m, n> = AsymptoticRing('QQ^m * m^QQ * n^ZZ * log(n)^ZZ', QQ)
sage: (4*2^m*m^4*log(n)).variable_names()
('m', 'n')
sage: (4*2^m*m^4).variable_names()
('m',)
sage: (4*log(n)).variable_names()
('n',)
sage: (4*m^3).variable_names()
('m',)
sage: (4*m^0).variable_names()
()
sage: (4*2^m*m^4 + log(n)).variable_names()
('m', 'n')
sage: (2^m + m^4 + log(n)).variable_names()
('m', 'n')
sage: (2^m + m^4).variable_names()
('m',)
```

class sage.rings.asymptotic.asymptotic_ring.**AsymptoticRing**(*growth_group*, *coefficient_ring*, *category*, *default_prec*, *term_monoid_factory*, *locals*)

Bases: sage.rings.ring.Algebra, sage.structure.unique_representation.UniqueRepresentation, *sage.rings.asymptotic.misc.WithLocals*

A ring consisting of *asymptotic expansions*.

INPUT:

- `growth_group` – either a partially ordered group (see *(Asymptotic) Growth Groups*) or a string describing such a growth group (see *GrowthGroupFactory*).
- `coefficient_ring` – the ring which contains the coefficients of the expansions.
- `default_prec` – a positive integer. This is the number of summands that are kept before truncating an infinite series.
- `category` – the category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of `Category of rings`. This is also the default category if `None` is specified.
- `term_monoid_factory` – a *TermMonoidFactory*. If `None`, then *DefaultTermMonoidFactory* is used.
- `locals` – a dictionary which may contain the following keys and values:

- 'log' – value: a function. If not given, then the usual log is taken. (See also *AsymptoticExpansion.log()*.)

EXAMPLES:

We begin with the construction of an asymptotic ring in various ways. First, we simply pass a string specifying the underlying growth group:

```
sage: R1_x.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=QQ); R1_x
Asymptotic Ring <x^QQ> over Rational Field
sage: x
x
```

This is equivalent to the following code, which explicitly specifies the underlying growth group:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G_QQ = GrowthGroup('x^QQ')
sage: R2_x.<x> = AsymptoticRing(growth_group=G_QQ, coefficient_ring=QQ); R2_x
Asymptotic Ring <x^QQ> over Rational Field
```

Of course, the coefficient ring of the asymptotic ring and the base ring of the underlying growth group do not need to coincide:

```
sage: R_ZZ_x.<x> = AsymptoticRing(growth_group='x^QQ', coefficient_ring=ZZ); R_ZZ_x
Asymptotic Ring <x^QQ> over Integer Ring
```

Note, we can also create and use logarithmic growth groups:

```
sage: R_log = AsymptoticRing(growth_group='log(x)^ZZ', coefficient_ring=QQ); R_log
Asymptotic Ring <log(x)^ZZ> over Rational Field
```

Other growth groups are available. See *Asymptotic Ring* for more examples.

Below there are some technical details.

According to the conventions for parents, uniqueness is ensured:

```
sage: R1_x is R2_x
True
```

Furthermore, the coercion framework is also involved. Coercion between two asymptotic rings is possible (given that the underlying growth groups and coefficient rings are chosen appropriately):

```
sage: R1_x.has_coerce_map_from(R_ZZ_x)
True
```

Additionally, for the sake of convenience, the coefficient ring also coerces into the asymptotic ring (representing constant quantities):

```
sage: R1_x.has_coerce_map_from(QQ)
True
```

It is possible to customize the terms in an asymptotic expansion:

```
sage: from sage.rings.asymptotic.term_monoid import ExactTermMonoid, OTermMonoid
sage: from sage.rings.asymptotic.term_monoid import TermMonoidFactory
sage: class MyExactTermMonoid(ExactTermMonoid):
```

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```
....:     pass
sage: class MyOTermMonoid(OTermMonoid):
....:     pass
sage: MyTermMonoid = TermMonoidFactory('MyTermMonoid',
....:                                  exact_term_monoid_class=MyExactTermMonoid,
....:                                  O_term_monoid_class=MyOTermMonoid)
sage: G = GrowthGroup('x^ZZ')
sage: A.<n> = AsymptoticRing(growth_group=G, coefficient_ring=QQ, term_monoid_
↪factory=MyTermMonoid)
sage: a = A.an_element(); a
1/8*x^3 + O(x)
sage: for t in a.summands.elements_topological(reverse=True):
....:     print(t, type(t))
1/8*x^3 <class '__main__.MyExactTermMonoid_with_category.element_class'>
O(x) <class '__main__.MyOTermMonoid_with_category.element_class'>
```

static B(*expression*, *valid_from=0*)

" Create a B-term.

INPUT:

- `valid_from` – dictionary mapping variable names to lower bounds for the corresponding variable. The bound implied by this term is valid when all variables are at least their corresponding lower bound. If a number is passed to `valid_from`, then the lower bounds for all variables of the asymptotic expansion are set to this number

OUTPUT:

A B-term

EXAMPLES:

```
sage: A.<x> = AsymptoticRing(growth_group='x^ZZ * QQ^y', coefficient_ring=QQ)
sage: A.B(2*x^3, {x: 5})
B(2*x^3, x >= 5)
```

Element

alias of *AsymptoticExpansion*

change_parameter(***kwds*)

Return an asymptotic ring with a change in one or more of the given parameters.

INPUT:

- `growth_group` – (default: `None`) the new growth group.
- `coefficient_ring` – (default: `None`) the new coefficient ring.
- `category` – (default: `None`) the new category.
- `default_prec` – (default: `None`) the new default precision.

OUTPUT:

An asymptotic ring.

EXAMPLES:

```
sage: A = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: A.change_parameter(coefficient_ring=QQ)
Asymptotic Ring <x^ZZ> over Rational Field
```

coefficient_ring

The coefficient ring of this asymptotic ring.

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.coefficient_ring
Integer Ring
```

coefficients_of_generating_function(*function*, *singularities*, *precision=None*, *return_singular_expansions=False*, *error_term=None*)

Return the asymptotic growth of the coefficients of some generating function by means of Singularity Analysis.

INPUT:

- `function` – a callable function in one variable.
- `singularities` – list of dominant singularities of the function.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.
- `return_singular_expansions` – (default: `False`) a boolean. If set, the singular expansions are also returned.
- `error_term` – (default: `None`) an asymptotic expansion. If `None`, then this is interpreted as zero. The contributions of the coefficients are added to `error_term` during Singularity Analysis.

OUTPUT:

- If `return_singular_expansions=False`: An asymptotic expansion from this ring.
- If `return_singular_expansions=True`: A named tuple with components `asymptotic_expansion` and `singular_expansions`. The former contains an asymptotic expansion from this ring, the latter is a dictionary which contains the singular expansions around the singularities.

Todo: Make this method more usable by implementing the processing of symbolic expressions.

EXAMPLES:

Catalan numbers:

```
sage: def catalan(z):
....:     return (1-(1-4*z)^(1/2))/(2*z)
sage: B.<n> = AsymptoticRing('QQ^n * n^QQ', QQ)
sage: B.coefficients_of_generating_function(catalan, (1/4,), precision=3)
1/sqrt(pi)*4^n*n^(-3/2) - 9/8/sqrt(pi)*4^n*n^(-5/2)
+ 145/128/sqrt(pi)*4^n*n^(-7/2) + O(4^n*n^(-4))
sage: B.coefficients_of_generating_function(catalan, (1/4,), precision=2,
....:                         return_singular_expansions=True)
SingularityAnalysisResult(asymptotic_expansion=1/sqrt(pi)*4^n*n^(-3/2)
```

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```
- 9/8/sqrt(pi)*4^n*n^(-5/2) + O(4^n*n^(-3)),
singular_expansions={1/4: 2 - 2*T^(-1/2)
+ 2*T^(-1) - 2*T^(-3/2) + O(T^(-2))})
```

Unit fractions:

```
sage: def logarithmic(z):
....:     return -log(1-z)
sage: B.coefficients_of_generating_function(logarithmic, (1,), precision=5)
n^(-1) + O(n^(-3))
```

Harmonic numbers:

```
sage: def harmonic(z):
....:     return -log(1-z)/(1-z)
sage: B.<n> = AsymptoticRing('QQ^n * n^QQ * log(n)^QQ', QQ)
sage: ex = B.coefficients_of_generating_function(harmonic, (1,), precision=13);␣
↪ex
log(n) + euler_gamma + 1/2*n^(-1) - 1/12*n^(-2) + 1/120*n^(-4)
+ O(n^(-6))
sage: ex.has_same_summands(asymptotic_expansions.HarmonicNumber(
....:     'n', precision=5))
True
```

Warning: Once singular expansions around points other than infinity are implemented (trac ticket #20050), the output in the case `return_singular_expansions` will change to return singular expansions around the singularities.

In the following example, the result is an exact asymptotic expression for sufficiently large n (i.e., there might be finitely many exceptional values). This is encoded by an $O(0)$ error term:

```
sage: def f(z):
....:     return z/(1-z)
sage: B.coefficients_of_generating_function(f, (1,), precision=3)
Traceback (most recent call last):
...
NotImplementedOZero: got 1 + O(0)
The error term O(0) means 0 for sufficiently large n.
```

In this case, we can manually intervene by adding an error term that suits us:

```
sage: B.coefficients_of_generating_function(f, (1,), precision=3,
....:                                       error_term=O(n^-100))
1 + O(n^(-100))
```

construction()

Return the construction of this asymptotic ring.

OUTPUT:

A pair whose first entry is an *asymptotic ring construction functor* and its second entry the coefficient ring.

EXAMPLES:

```
sage: A = AsymptoticRing(growth_group='x^ZZ * QQ^y', coefficient_ring=QQ)
sage: A.construction()
(AsymptoticRing<x^ZZ * QQ^y * Signs^y>, Rational Field)
```

See also:

Asymptotic Ring, *AsymptoticRing*, *AsymptoticRingFunctor*.

create_summand(*type*, *data=None*, ***kwds*)

Create a simple asymptotic expansion consisting of a single summand.

INPUT:

- `type` – 'O' or 'exact'.
- `data` – the element out of which a summand has to be created.
- `growth` – an element of the *growth_group()*.
- `coefficient` – an element of the *coefficient_ring()*.

Note: Either `growth` and `coefficient` or `data` have to be specified.

OUTPUT:

An asymptotic expansion.

Note: This method calls the factory *TermMonoid* with the appropriate arguments.

EXAMPLES:

```
sage: R = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: R.create_summand('O', x^2)
O(x^2)
sage: R.create_summand('exact', growth=x^456, coefficient=123)
123*x^456
sage: R.create_summand('exact', data=12*x^13)
12*x^13
```

default_prec

The default precision of this asymptotic ring.

This is the parameter used to determine how many summands are kept before truncating an infinite series (which occur when inverting asymptotic expansions).

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.default_prec
20
sage: AR = AsymptoticRing('x^ZZ', ZZ, default_prec=123)
sage: AR.default_prec
123
```

gen(*n=0*)

Return the `n`-th generator of this asymptotic ring.

INPUT:

- n – (default: 0) a non-negative integer.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: R.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: R.gen()
x
```

gens()

Return a tuple with generators of this asymptotic ring.

INPUT:

Nothing.

OUTPUT:

A tuple of asymptotic expansions.

Note: Generators do not necessarily exist. This depends on the underlying growth group. For example, *monomial growth groups* have a generator, and exponential growth groups do not.

EXAMPLES:

```
sage: AR.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.gens()
(x,)
sage: B.<y,z> = AsymptoticRing(growth_group='y^ZZ * z^ZZ', coefficient_ring=QQ)
sage: B.gens()
(y, z)
```

growth_group

The growth group of this asymptotic ring.

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.growth_group
Growth Group x^ZZ
```

See also:

(Asymptotic) Growth Groups

ngens()

Return the number of generators of this asymptotic ring.

INPUT:

Nothing.

OUTPUT:

An integer.

EXAMPLES:

```
sage: AR.<x> = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.ngens()
1
```

some_elements()

Return some elements of this term monoid.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from itertools import islice
sage: A = AsymptoticRing(growth_group='z^QQ', coefficient_ring=ZZ)
sage: tuple(islice(A.some_elements(), int(10)))
(z^(3/2) + O(z^(1/2)),
 O(z^(1/2)),
 z^(3/2) + O(z^(-1/2)),
 -z^(3/2) + O(z^(1/2)),
 O(z^(-1/2)),
 O(z^2),
 z^6 + O(z^(1/2)),
 -z^(3/2) + O(z^(-1/2)),
 O(z^2),
 z^(3/2) + O(z^(-2)))
```

term_monoid(*type*)

Return the term monoid of this asymptotic ring of specified type.

INPUT:

- type – 'O' or 'exact', or an instance of an existing term monoid. See *TermMonoidFactory* for more details.

OUTPUT:

A term monoid object derived from *GenericTermMonoid*.

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.term_monoid('exact')
Exact Term Monoid x^ZZ with coefficients in Integer Ring
sage: AR.term_monoid('O')
O-Term Monoid x^ZZ with implicit coefficients in Integer Ring
sage: AR.term_monoid(AR.term_monoid('exact'))
Exact Term Monoid x^ZZ with coefficients in Integer Ring
```

term_monoid_factory

The term monoid factory of this asymptotic ring.

EXAMPLES:

```
sage: AR = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=ZZ)
sage: AR.term_monoid_factory
Term Monoid Factory 'sage.rings.asymptotic.term_monoid.DefaultTermMonoidFactory'
```

See also:

(Asymptotic) Term Monoids

variable_names()

Return the names of the variables.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: A = AsymptoticRing(growth_group='x^ZZ * QQ^y', coefficient_ring=QQ)
sage: A.variable_names()
('x', 'y')
```

class sage.rings.asymptotic.asymptotic_ring.**AsymptoticRingFunctor**(*growth_group*, *default_prec=None*, *category=None*, *term_monoid_factory=None*, *locals=None*, *cls=None*)

Bases: sage.categories.pushout.ConstructionFunctor

A construction functor for *asymptotic rings*.

INPUT:

- growth_group – a partially ordered group (see *AsymptoticRing* or *(Asymptotic) Growth Groups* for details).
- default_prec – None (default) or an integer.
- category – None (default) or a category.
- cls – *AsymptoticRing* (default) or a derived class.

EXAMPLES:

```
sage: AsymptoticRing(growth_group='x^ZZ', coefficient_ring=QQ).construction()  #␣
↪indirect doctest
(AsymptoticRing<x^ZZ>, Rational Field)
```

See also:

Asymptotic Ring, *AsymptoticRing*, *sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor*, *sage.rings.asymptotic.growth_group.ExponentialGrowthGroupFunctor*, *sage.rings.asymptotic.growth_group.MonomialGrowthGroupFunctor*, sage.categories.pushout.ConstructionFunctor.

merge(*other*)

Merge this functor with other if possible.

INPUT:

- other – a functor.

OUTPUT:

A functor or None.

EXAMPLES:

```
sage: X = AsymptoticRing(growth_group='x^ZZ', coefficient_ring=QQ)
sage: Y = AsymptoticRing(growth_group='y^ZZ', coefficient_ring=QQ)
sage: F_X = X.construction()[0]
sage: F_Y = Y.construction()[0]
sage: F_X.merge(F_X)
AsymptoticRing<x^ZZ>
sage: F_X.merge(F_Y)
AsymptoticRing<x^ZZ * y^ZZ>
```

exception sage.rings.asymptotic.asymptotic_ring.**NoConvergenceError**

Bases: RuntimeError

A special RuntimeError which is raised when an algorithm does not converge/stop.

4.2 Common Asymptotic Expansions

Asymptotic expansions in SageMath can be built through the `asymptotic_expansions` object. It contains generators for common asymptotic expressions. For example,

```
sage: asymptotic_expansions.Stirling('n', precision=5)
sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(1/2) +
1/12*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-1/2) +
1/288*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-3/2) +
O(e^(n*log(n))*(e^n)^(-1)*n^(-5/2))
```

generates the first 5 summands of Stirling's approximation formula for factorials.

To construct an asymptotic expression manually, you can use the class *AsymptoticRing*. See *asymptotic ring* for more details and a lot of examples.

Asymptotic Expansions

HarmonicNumber()	harmonic numbers
Stirling()	Stirling's approximation formula for factorials
log_Stirling()	the logarithm of Stirling's approximation formula for factorials
Binomial_kn_over_n()	an asymptotic expansion of the binomial coefficient
SingularityAnalysis()	an asymptotic expansion obtained by singularity analysis
ImplicitExpansion()	the singular expansion of a function $y(z)$ satisfying $y(z) = z\Phi(y(z))$
ImplicitExpansionPeriodicPart()	the singular expansion of the periodic part of a function $y(z)$ satisfying $y(z) = z\Phi(y(z))$
InverseFunctionAnalysis()	coefficient growth of a function $y(z)$ defined implicitly by $y(z) = z\Phi(y(z))$

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4.2.1 Classes and Methods

class sage.rings.asymptotic.asymptotic_expansion_generators.**AsymptoticExpansionGenerators**

Bases: sage.structure.sage_object.SageObject

A collection of constructors for several common asymptotic expansions.

A list of all asymptotic expansions in this database is available via tab completion. Type "`asymptotic_expansions.`" and then hit tab to see which expansions are available.

The asymptotic expansions currently in this class include:

- *HarmonicNumber()*
- *Stirling()*
- *log_Stirling()*
- *Binomial_kn_over_n()*
- *SingularityAnalysis()*
- *ImplicitExpansion()*
- *ImplicitExpansionPeriodicPart()*
- *InverseFunctionAnalysis()*

static Binomial_kn_over_n(*var*, *k*, *precision=None*, *skip_constant_factor=False*)

Return the asymptotic expansion of the binomial coefficient kn choose n.

INPUT:

- `var` – a string for the variable name.
- `k` – a number or symbolic constant.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.
- `skip_constant_factor` – (default: `False`) a boolean. If set, then the constant factor $\sqrt{k/(2\pi(k-1))}$ is left out. As a consequence, the coefficient ring of the output changes from `Symbolic Constants Subring` (if `False`) to `Rational Field` (if `True`).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.Binomial_kn_over_n('n', k=2, precision=3)
1/sqrt(pi)*4^n*n^(-1/2)
- 1/8/sqrt(pi)*4^n*n^(-3/2)
+ 1/128/sqrt(pi)*4^n*n^(-5/2)
+ O(4^n*n^(-7/2))
sage: _.parent()
Asymptotic Ring <QQ^n * n^QQ> over Symbolic Constants Subring
```

```
sage: asymptotic_expansions.Binomial_kn_over_n('n', k=3, precision=3)
1/2*sqrt(3)/sqrt(pi)*(27/4)^n*n^(-1/2)
- 7/144*sqrt(3)/sqrt(pi)*(27/4)^n*n^(-3/2)
+ 49/20736*sqrt(3)/sqrt(pi)*(27/4)^n*n^(-5/2)
+ O((27/4)^n*n^(-7/2))
```

```
sage: asymptotic_expansions.Binomial_kn_over_n('n', k=7/5, precision=3)
1/2*sqrt(7)/sqrt(pi)*(7/10*7^(2/5)*2^(3/5))^n*n^(-1/2)
- 13/112*sqrt(7)/sqrt(pi)*(7/10*7^(2/5)*2^(3/5))^n*n^(-3/2)
+ 169/12544*sqrt(7)/sqrt(pi)*(7/10*7^(2/5)*2^(3/5))^n*n^(-5/2)
+ O((7/10*7^(2/5)*2^(3/5))^n*n^(-7/2))
sage: _.parent()
Asymptotic Ring <(Symbolic Constants Subring)^n * n^QQ>
over Symbolic Constants Subring
```

static HarmonicNumber(*var*, *precision=None*, *skip_constant_summand=False*)

Return the asymptotic expansion of a harmonic number.

INPUT:

- `var` – a string for the variable name.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.
- `skip_constant_summand` – (default: `False`) a boolean. If set, then the constant summand `euler_gamma` is left out. As a consequence, the coefficient ring of the output changes from `Symbolic Constants Subring` (if `False`) to `Rational Field` (if `True`).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.HarmonicNumber('n', precision=5)
log(n) + euler_gamma + 1/2*n^(-1) - 1/12*n^(-2) + 1/120*n^(-4) + O(n^(-6))
```

static ImplicitExpansion(*var*, *phi*, *tau=None*, *precision=None*)

Return the singular expansion for a function $y(z)$ defined implicitly by $y(z) = z\Phi(y(z))$.

The function Φ is assumed to be analytic around 0. Furthermore, Φ is not allowed to be an affine-linear function and we require $\Phi(0) \neq 0$.

Furthermore, it is assumed that there is a unique positive solution τ of $\Phi(\tau) - \tau\Phi'(\tau) = 0$.

All details are given in Chapter VI.7 of [FS2009].

INPUT:

- `var` – a string for the variable name.
- `phi` – the function Φ. See the extended description for assumptions on Φ.
- `tau` – (default: `None`) the fundamental constant described in the extended description. If `None`, then τ is determined automatically if possible.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.

OUTPUT:

An asymptotic expansion.

Note: In the given case, the radius of convergence of the function of interest is known to be $\rho = \tau/\Phi(\tau)$. Until trac ticket #20050 is implemented, the variable in the returned asymptotic expansion represents a singular element of the form $(1 - z/\rho)^{-1}$, for the variable $z \to \rho$.

EXAMPLES:

We can, for example, determine the singular expansion of the well-known tree function T (which satisfies $T(z) = z\exp(T(z))$):

```
sage: asymptotic_expansions.ImplicitExpansion('Z', phi=exp, precision=8)
doctest:warning
...
FutureWarning: This class/method/function is marked as experimental. It, its␣
↪functionality or its interface might change without a formal deprecation.
See http://trac.sagemath.org/20050 for details.
1 - sqrt(2)*Z^(-1/2) + 2/3*Z^(-1) - 11/36*sqrt(2)*Z^(-3/2) +
43/135*Z^(-2) - 769/4320*sqrt(2)*Z^(-5/2) + 1768/8505*Z^(-3) + O(Z^(-7/2))
```

Another classical example in this context is the generating function $B(z)$ enumerating binary trees with respect to the number of inner nodes. The function satisfies $B(z) = z(1+2B(z)+B(z)^2)$, which can also be solved explicitly, yielding $B(z) = \frac{1-\sqrt{1-4z}}{2z} - 1$. We compare the expansions from both approaches:

```
sage: def B(z):
....:     return (1 - sqrt(1 - 4*z))/(2*z) - 1
sage: A.<Z> = AsymptoticRing('Z^QQ', QQ, default_prec=3)
sage: B((1-1/Z)/4)
1 - 2*Z^(-1/2) + 2*Z^(-1) - 2*Z^(-3/2) + 2*Z^(-2)
- 2*Z^(-5/2) + O(Z^(-3))
sage: asymptotic_expansions.ImplicitExpansion(Z, phi=lambda u: 1 + 2*u + u^2,␣
↪precision=7)
1 - 2*Z^(-1/2) + 2*Z^(-1) - 2*Z^(-3/2) + 2*Z^(-2)
- 2*Z^(-5/2) + O(Z^(-3))
```

Neither τ nor Φ have to be known explicitly, they can also be passed symbolically:

```
sage: tau = var('tau')
sage: phi = function('phi')
sage: asymptotic_expansions.ImplicitExpansion('Z', phi=phi, tau=tau,␣
↪precision=3)  # long time
tau + (-sqrt(2)*sqrt(-tau*phi(tau)^2/(2*tau*diff(phi(tau), tau)^2
- tau*phi(tau)*diff(phi(tau), tau, tau)
- 2*phi(tau)*diff(phi(tau), tau))))*Z^(-1/2) + O(Z^(-1))
```

Note that we do not check whether a passed τ actually satisfies the requirements. Only the first of the following expansions is correct:

```
sage: asymptotic_expansions.ImplicitExpansion('Z',
....:     phi=lambda u: 1 + 2*u + u^2, precision=5) # correct expansion
1 - 2*Z^(-1/2) + 2*Z^(-1) - 2*Z^(-3/2) + O(Z^(-2))
sage: asymptotic_expansions.ImplicitExpansion('Z', phi=lambda u: 1 + 2*u + u^2,␣
↪tau=2, precision=5)
```

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```
Traceback (most recent call last):
...
ZeroDivisionError: symbolic division by zero
sage: asymptotic_expansions.ImplicitExpansion('Z', phi=lambda u: 1 + 2*u + u^2,␣
↪tau=3, precision=5)
3 - 4*I*sqrt(3)*Z^(-1/2) + 6*I*sqrt(3)*Z^(-3/2) + O(Z^(-2))
```

See also:

ImplicitExpansionPeriodicPart(), *InverseFunctionAnalysis()*.

static ImplicitExpansionPeriodicPart(*var*, *phi*, *period*, *tau=None*, *precision=None*)

Return the singular expansion for the periodic part of a function $y(z)$ defined implicitly by $y(z) = z\Phi(y(z))$.

The function Φ is assumed to be analytic around 0. Furthermore, Φ is not allowed to be an affine-linear function and we require $\Phi(0) \neq 0$. For an integer p, Φ is called p-periodic if we have $\Psi(u^p) = \Phi(u)$ for a power series Ψ where p is maximal.

Furthermore, it is assumed that there is a unique positive solution τ of $\Phi(\tau) - \tau\Phi'(\tau) = 0$.

If Φ is p-periodic, then we have $y(z) = zg(z^p)$. This method returns the singular expansion of $g(z)$.

INPUT:

- `var` – a string for the variable name.
- `phi` – the function Φ. See the extended description for assumptions on Φ.
- `period` – the period of the function Φ. See the extended description for details.
- `tau` – (default: `None`) the fundamental constant described in the extended description. If `None`, then τ is determined automatically if possible.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.

OUTPUT:

An asymptotic expansion.

Note: In the given case, the radius of convergence of the function of interest is known to be $\rho = \tau/\Phi(\tau)$. Until trac ticket #20050 is implemented, the variable in the returned asymptotic expansion represents a singular element of the form $(1 - z/\rho)^{-1}$, for the variable $z \to \rho$.

See also:

ImplicitExpansion(), *InverseFunctionAnalysis()*.

EXAMPLES:

The generating function enumerating binary trees with respect to tree size satisfies $B(z) = z(1 + B(z)^2)$. This function can be written as $B(z) = zg(z^2)$, and as $B(z)$ can be determined explicitly we have $g(z) = \frac{1-\sqrt{1-4z}}{2z}$. We compare the corresponding expansions:

```
sage: asymptotic_expansions.ImplicitExpansionPeriodicPart('Z', phi=lambda u: 1␣
↪+ u^2,
....:                                                    period=2, precision=7)
doctest:warning
```

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```
...
FutureWarning: This class/method/function is marked as experimental. It, its␣
↪functionality or its interface might change without a formal deprecation.
See http://trac.sagemath.org/20050 for details.
2 - 2*Z^(-1/2) + 2*Z^(-1) - 2*Z^(-3/2) + 2*Z^(-2) - 2*Z^(-5/2) + O(Z^(-3))
sage: def g(z):
....:     return (1 - sqrt(1 - 4*z))/(2*z)
sage: A.<Z> = AsymptoticRing('Z^QQ', QQ, default_prec=3)
sage: g((1 - 1/Z)/4)
2 - 2*Z^(-1/2) + 2*Z^(-1) - 2*Z^(-3/2) + 2*Z^(-2) - 2*Z^(-5/2) + O(Z^(-3))
```

static InverseFunctionAnalysis(*var*, *phi*, *tau=None*, *period=1*, *precision=None*)

Return the coefficient growth of a function $y(z)$ defined implicitly by $y(z) = z\Phi(y(z))$.

The function Φ is assumed to be analytic around 0. Furthermore, Φ is not allowed to be an affine-linear function and we require $\Phi(0) \neq 0$. For an integer p, Φ is called p-periodic if we have $\Psi(u^p) = \Phi(u)$ for a power series Ψ where p is maximal.

Furthermore, it is assumed that there is a unique positive solution τ of $\Phi(\tau) - \tau\Phi'(\tau) = 0$.

INPUT:

- `var` – a string for the variable name.
- `phi` – the function Φ. See the extended description for assumptions on Φ.
- `tau` – (default: `None`) the fundamental constant described in the extended description. If `None`, then τ is determined automatically if possible.
- `period` – (default: 1) the period of the function Φ. See the extended description for details.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.

OUTPUT:

An asymptotic expansion.

Note: It is not checked that the passed period actually fits to the passed function Φ.

The resulting asymptotic expansion is only valid for $n \equiv 1 \mod p$, where p is the period. All other coefficients are 0.

EXAMPLES:

There are C_n (the n-th Catalan number) different binary trees of size $2n+1$, and there are no binary trees with an even number of nodes. The corresponding generating function satisfies $B(z) = z(1 + B(z)^2)$, which allows us to compare the asymptotic expansions for the number of binary trees of size n obtained via C_n and obtained via the analysis of $B(z)$:

```
sage: assume(SR.an_element() > 0)
sage: A.<n> = AsymptoticRing('QQ^n * n^QQ', SR)
sage: binomial_expansion = asymptotic_expansions.Binomial_kn_over_n(n, k=2,␣
↪precision=3)
sage: catalan_expansion = binomial_expansion / (n+1)
sage: catalan_expansion.subs(n=(n-1)/2)
2*sqrt(1/2)/sqrt(pi)*2^n*n^(-3/2) - 3/2*sqrt(1/2)/sqrt(pi)*2^n*n^(-5/2)
```

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```
+ 25/16*sqrt(1/2)/sqrt(pi)*2^n*n^(-7/2) + O(2^n*n^(-9/2))
sage: asymptotic_expansions.InverseFunctionAnalysis(n, phi=lambda u: 1 + u^2,␣
↪period=2,
....:                                                tau=1, precision=8)
2*sqrt(1/2)/sqrt(pi)*2^n*n^(-3/2) - 3/2*sqrt(1/2)/sqrt(pi)*2^n*n^(-5/2)
+ 25/16*sqrt(1/2)/sqrt(pi)*2^n*n^(-7/2) + O(2^n*n^(-9/2))
```

The code in the aperiodic case is more efficient, however. Therefore, it is recommended to use combinatorial identities to reduce to the aperiodic case. In the example above, this is well-known: we now count binary trees with n internal nodes. The corresponding generating function satisfies $B(z) = z(1+2B(z)+B(z)^2)$:

```
sage: catalan_expansion
1/sqrt(pi)*4^n*n^(-3/2) - 9/8/sqrt(pi)*4^n*n^(-5/2)
+ 145/128/sqrt(pi)*4^n*n^(-7/2) + O(4^n*n^(-9/2))
sage: asymptotic_expansions.InverseFunctionAnalysis(n, phi=lambda u: 1 + 2*u +␣
↪u^2,
....:                                                tau=1, precision=8)
1/sqrt(pi)*4^n*n^(-3/2) - 9/8/sqrt(pi)*4^n*n^(-5/2)
+ 145/128/sqrt(pi)*4^n*n^(-7/2) + O(4^n*n^(-9/2))

sage: forget()
```

See also:

ImplicitExpansion(), *ImplicitExpansionPeriodicPart()*.

static SingularityAnalysis(*var*, *zeta=1*, *alpha=0*, *beta=0*, *delta=0*, *precision=None*, *normalized=True*)

Return the asymptotic expansion of the coefficients of an power series with specified pole and logarithmic singularity.

More precisely, this extracts the n-th coefficient

$$[z^n]\left(\frac{1}{1-z/\zeta}\right)^{\alpha}\left(\frac{1}{z/\zeta}\log\frac{1}{1-z/\zeta}\right)^{\beta}\left(\frac{1}{z/\zeta}\log\left(\frac{1}{z/\zeta}\log\frac{1}{1-z/\zeta}\right)\right)^{\delta}$$

(if `normalized=True`, the default) or

$$[z^n]\left(\frac{1}{1-z/\zeta}\right)^{\alpha}\left(\log\frac{1}{1-z/\zeta}\right)^{\beta}\left(\log\left(\frac{1}{z/\zeta}\log\frac{1}{1-z/\zeta}\right)\right)^{\delta}$$

(if `normalized=False`).

INPUT:

- `var` – a string for the variable name.
- `zeta` – (default: 1) the location of the singularity.
- `alpha` – (default: 0) the pole order of the singularity.
- `beta` – (default: 0) the order of the logarithmic singularity.
- `delta` – (default: 0) the order of the log-log singularity. Not yet implemented for `delta != 0`.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.
- `normalized` – (default: `True`) a boolean, see above.

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=1)
1
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=2)
n + 1
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=3)
1/2*n^2 + 3/2*n + 1
sage: _.parent()
Asymptotic Ring <n^ZZ> over Rational Field
```

```
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=-3/2,
....:     precision=3)
3/4/sqrt(pi)*n^(-5/2)
+ 45/32/sqrt(pi)*n^(-7/2)
+ 1155/512/sqrt(pi)*n^(-9/2)
+ O(n^(-11/2))
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=-1/2,
....:     precision=3)
-1/2/sqrt(pi)*n^(-3/2)
- 3/16/sqrt(pi)*n^(-5/2)
- 25/256/sqrt(pi)*n^(-7/2)
+ O(n^(-9/2))
sage: asymptotic_expansions.SingularityAnalysis('n', alpha=1/2,
....:     precision=4)
1/sqrt(pi)*n^(-1/2)
- 1/8/sqrt(pi)*n^(-3/2)
+ 1/128/sqrt(pi)*n^(-5/2)
+ 5/1024/sqrt(pi)*n^(-7/2)
+ O(n^(-9/2))
sage: _.parent()
Asymptotic Ring <n^QQ> over Symbolic Constants Subring
```

```
sage: S = SR.subring(rejecting_variables=('n',))
sage: asymptotic_expansions.SingularityAnalysis(
....:     'n', alpha=S.var('a'),
....:     precision=4).map_coefficients(lambda c: c.factor())
1/gamma(a)*n^(a - 1)
+ (1/2*(a - 1)*a/gamma(a))*n^(a - 2)
+ (1/24*(3*a - 1)*(a - 1)*(a - 2)*a/gamma(a))*n^(a - 3)
+ (1/48*(a - 1)^2*(a - 2)*(a - 3)*a^2/gamma(a))*n^(a - 4)
+ O(n^(a - 5))
sage: _.parent()
Asymptotic Ring <n^(Symbolic Subring rejecting the variable n)>
over Symbolic Subring rejecting the variable n
```

```
sage: ae = asymptotic_expansions.SingularityAnalysis('n',
....:           alpha=1/2, beta=1, precision=4); ae
1/sqrt(pi)*n^(-1/2)*log(n) + ((euler_gamma + 2*log(2))/sqrt(pi))*n^(-1/2)
- 5/8/sqrt(pi)*n^(-3/2)*log(n) + (1/8*(3*euler_gamma + 6*log(2) - 8)/sqrt(pi)
```

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```
- (euler_gamma + 2*log(2) - 2)/sqrt(pi))*n^(-3/2) + O(n^(-5/2)*log(n))
sage: n = ae.parent().gen()
sage: ae.subs(n=n-1).map_coefficients(lambda x: x.canonicalize_radical())
1/sqrt(pi)*n^(-1/2)*log(n)
+ ((euler_gamma + 2*log(2))/sqrt(pi))*n^(-1/2)
- 1/8/sqrt(pi)*n^(-3/2)*log(n)
+ (-1/8*(euler_gamma + 2*log(2))/sqrt(pi))*n^(-3/2)
+ O(n^(-5/2)*log(n))
```

```
sage: asymptotic_expansions.SingularityAnalysis('n',
....:     alpha=1, beta=1/2, precision=4)
log(n)^(1/2) + 1/2*euler_gamma*log(n)^(-1/2)
+ (-1/8*euler_gamma^2 + 1/48*pi^2)*log(n)^(-3/2)
+ (1/16*euler_gamma^3 - 1/32*euler_gamma*pi^2 + 1/8*zeta(3))*log(n)^(-5/2)
+ O(log(n)^(-7/2))
```

```
sage: ae = asymptotic_expansions.SingularityAnalysis('n',
....:     alpha=0, beta=2, precision=14)
sage: n = ae.parent().gen()
sage: ae.subs(n=n-2)
2*n^(-1)*log(n) + 2*euler_gamma*n^(-1) - n^(-2) - 1/6*n^(-3) + O(n^(-5))
```

```
sage: asymptotic_expansions.SingularityAnalysis(
....:     'n', 1, alpha=-1/2, beta=1, precision=2, normalized=False)
-1/2/sqrt(pi)*n^(-3/2)*log(n)
+ (-1/2*(euler_gamma + 2*log(2) - 2)/sqrt(pi))*n^(-3/2)
+ O(n^(-5/2)*log(n))
sage: asymptotic_expansions.SingularityAnalysis(
....:     'n', 1/2, alpha=0, beta=1, precision=3, normalized=False)
2^n*n^(-1) + O(2^n*n^(-2))
```

ALGORITHM:

See [FS2009].

static Stirling(*var*, *precision=None*, *skip_constant_factor=False*)

Return Stirling's approximation formula for factorials.

INPUT:

- `var` – a string for the variable name.
- `precision` – (default: `None`) an integer ≥ 3. If `None`, then the default precision of the asymptotic ring is used.
- `skip_constant_factor` – (default: `False`) a boolean. If set, then the constant factor $\sqrt{2\pi}$ is left out. As a consequence, the coefficient ring of the output changes from `Symbolic Constants Subring` (if `False`) to `Rational Field` (if `True`).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.Stirling('n', precision=5)
sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(1/2) +
1/12*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-1/2) +
1/288*sqrt(2)*sqrt(pi)*e^(n*log(n))*(e^n)^(-1)*n^(-3/2) +
O(e^(n*log(n))*(e^n)^(-1)*n^(-5/2))
sage: _.parent()
Asymptotic Ring <(e^(n*log(n)))^QQ * (e^n)^QQ * n^QQ * log(n)^QQ>
over Symbolic Constants Subring
```

See also:

log_Stirling(), *factorial()*.

static log_Stirling(*var*, *precision=None*, *skip_constant_summand=False*)

Return the logarithm of Stirling's approximation formula for factorials.

INPUT:

- `var` – a string for the variable name.
- `precision` – (default: `None`) an integer. If `None`, then the default precision of the asymptotic ring is used.
- `skip_constant_summand` – (default: `False`) a boolean. If set, then the constant summand $\log(2\pi)/2$ is left out. As a consequence, the coefficient ring of the output changes from `Symbolic Constants Subring` (if `False`) to `Rational Field` (if `True`).

OUTPUT:

An asymptotic expansion.

EXAMPLES:

```
sage: asymptotic_expansions.log_Stirling('n', precision=7)
n*log(n) - n + 1/2*log(n) + 1/2*log(2*pi) + 1/12*n^(-1)
- 1/360*n^(-3) + 1/1260*n^(-5) + O(n^(-7))
```

See also:

Stirling(), *factorial()*.

sage.rings.asymptotic.asymptotic_expansion_generators.asymptotic_expansions = <sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators object>

A collection of several common asymptotic expansions.

This is an instance of *AsymptoticExpansionGenerators* whose documentation provides more details.

4.3 (Asymptotic) Growth Groups

This module provides support for (asymptotic) growth groups.

Such groups are equipped with a partial order: the elements can be seen as functions, and the behavior as their argument (or arguments) gets large (tend to ∞) is compared.

Growth groups are used for the calculations done in the *asymptotic ring*. There, take a look at the *informal definition*, where examples of growth groups and elements are given as well.

4.3.1 Description of Growth Groups

Many growth groups can be described by a string, which can also be used to create them. For example, the string `'x^QQ * log(x)^ZZ * QQ^y * y^QQ'` represents a growth group with the following properties:

- It is a growth group in the two variables x and y.
- Its elements are of the form

$$x^r \cdot \log(x)^s \cdot a^y \cdot y^q$$

  for $r \in \mathbf{Q}$, $s \in \mathbf{Z}$, $a \in \mathbf{Q}$ and $q \in \mathbf{Q}$.
- The order is with respect to $x \to \infty$ and $y \to \infty$ independently of each other.
- To compare such elements, they are split into parts belonging to only one variable. In the example above,

$$x^{r_1} \cdot \log(x)^{s_1} \leq x^{r_2} \cdot \log(x)^{s_2}$$

  if $(r_1, s_1) \leq (r_2, s_2)$ lexicographically. This reflects the fact that elements x^r are larger than elements $\log(x)^s$ as $x \to \infty$. The factors belonging to the variable y are compared analogously.

  The results of these comparisons are then put together using the product order, i.e., $\leq$ if each component satisfies $\leq$.

Each description string consists of ordered factors—yes, this means * is noncommutative—of strings describing "elementary" growth groups (see the examples below). As stated in the example above, these factors are split by their variable; factors with the same variable are grouped. Reading such factors from left to right determines the order: Comparing elements of two factors (growth groups) L and R, then all elements of L are considered to be larger than each element of R.

4.3.2 Creating a Growth Group

For many purposes the factory `GrowthGroup` (see *GrowthGroupFactory*) is the most convenient way to generate a growth group.

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
```

Here are some examples:

```
sage: GrowthGroup('z^ZZ')
Growth Group z^ZZ
sage: M = GrowthGroup('z^QQ'); M
Growth Group z^QQ
```

Each of these two generated groups is a *MonomialGrowthGroup*, whose elements are powers of a fixed symbol (above `'z'`). For the order of the elements it is assumed that $z \to \infty$.

Note: Growth groups where the variable tend to some value distinct from ∞ are not yet implemented.

To create elements of M, a generator can be used:

```
sage: z = M.gen()
sage: z^(3/5)
z^(3/5)
```

Strings can also be parsed:

```
sage: M('z^7')
z^7
```

Similarly, we can construct logarithmic factors by:

```
sage: GrowthGroup('log(z)^QQ')
Growth Group log(z)^QQ
```

which again creates a *MonomialGrowthGroup*. An *ExponentialGrowthGroup* is generated in the same way. Our factory gives

```
sage: E = GrowthGroup('(QQ_+)^z'); E
Growth Group QQ^z
```

and a typical element looks like this:

```
sage: E.an_element()
(1/2)^z
```

More complex groups are created in a similar fashion. For example

```
sage: C = GrowthGroup('(QQ_+)^z * z^QQ * log(z)^QQ'); C
Growth Group QQ^z * z^QQ * log(z)^QQ
```

This contains elements of the form

```
sage: C.an_element()
(1/2)^z*z^(1/2)*log(z)^(1/2)
```

The group C itself is a Cartesian product; to be precise a *UnivariateProduct*. We can see its factors:

```
sage: C.cartesian_factors()
(Growth Group QQ^z, Growth Group z^QQ, Growth Group log(z)^QQ)
```

Multivariate constructions are also possible:

```
sage: GrowthGroup('x^QQ * y^QQ')
Growth Group x^QQ * y^QQ
```

This gives a *MultivariateProduct*.

Both these Cartesian products are derived from the class *GenericProduct*. Moreover all growth groups have the abstract base class *GenericGrowthGroup* in common.

Some Examples

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G_x = GrowthGroup('x^ZZ'); G_x
Growth Group x^ZZ
sage: G_xy = GrowthGroup('x^ZZ * y^ZZ'); G_xy
Growth Group x^ZZ * y^ZZ
sage: G_xy.an_element()
x*y
sage: x = G_xy('x'); y = G_xy('y')
sage: x^2
x^2
sage: elem = x^21*y^21; elem^2
x^42*y^42
```

A monomial growth group itself is totally ordered, all elements are comparable. However, this does **not** hold for Cartesian products:

```
sage: e1 = x^2*y; e2 = x*y^2
sage: e1 <= e2 or e2 <= e1
False
```

In terms of uniqueness, we have the following behaviour:

```
sage: GrowthGroup('x^ZZ * y^ZZ') is GrowthGroup('y^ZZ * x^ZZ')
True
```

The above is `True` since the order of the factors does not play a role here; they use different variables. But when using the same variable, it plays a role:

```
sage: GrowthGroup('x^ZZ * log(x)^ZZ') is GrowthGroup('log(x)^ZZ * x^ZZ')
False
```

In this case the components are ordered lexicographically, which means that in the second growth group, `log(x)` is assumed to grow faster than `x` (which is nonsense, mathematically). See *CartesianProduct* for more details or see *above* for a more extensive description.

Short notation also allows the construction of more complicated growth groups:

```
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ * log(x)^QQ * y^QQ')
sage: G.an_element()
(1/2)^x*x*log(x)^(1/2)*y^(1/2)
sage: x, y = var('x y')
sage: G(2^x * log(x) * y^(1/2)) * G(x^(-5) * 5^x * y^(1/3))
10^x*x^(-5)*log(x)*y^(5/6)
```

AUTHORS:

- Benjamin Hackl (2015)
- Daniel Krenn (2015)
- Clemens Heuberger (2016)

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4.3.3 Classes and Methods

class sage.rings.asymptotic.growth_group.**AbstractGrowthGroupFunctor**(*var*, *domain*)

Bases: sage.categories.pushout.ConstructionFunctor

A base class for the functors constructing growth groups.

INPUT:

- `var` – a string or list of strings (or anything else *Variable* accepts).
- `domain` – a category.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('z^QQ').construction()[0]  # indirect doctest
MonomialGrowthGroup[z]
```

See also:

Asymptotic Ring, *ExponentialGrowthGroupFunctor*, *MonomialGrowthGroupFunctor*, *sage.rings.asymptotic.asymptotic_ring.AsymptoticRingFunctor*, sage.categories.pushout.ConstructionFunctor.

merge(*other*)

Merge this functor with `other` of possible.

INPUT:

- `other` – a functor.

OUTPUT:

A functor or `None`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: F = GrowthGroup('(QQ_+)^t').construction()[0]
sage: G = GrowthGroup('t^QQ').construction()[0]
sage: F.merge(F)
ExponentialGrowthGroup[t]
sage: F.merge(G) is None
True
```

exception sage.rings.asymptotic.growth_group.**DecreasingGrowthElementError**(*element*, **args*, ***kwds*)

Bases: ValueError

A special ValueError which is raised when a growth element is less than one.

INPUT:

- `element` – a *GenericGrowthElement*

The remaining arguments are passed on to ValueError.

class sage.rings.asymptotic.growth_group.**ExponentialGrowthElement**(*parent*, *raw_element*)
Bases: *sage.rings.asymptotic.growth_group.GenericGrowthElement*

An implementation of exponential growth elements.

INPUT:

- `parent` – an *ExponentialGrowthGroup*.
- `raw_element` – an element from the base ring of the parent.

  This `raw_element` is the base of the created exponential growth element.

An exponential growth element represents a term of the type $\mathrm{base}^{\mathrm{variable}}$. The multiplication corresponds to the multiplication of the bases.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('(ZZ_+)^x')
sage: e1 = P(1); e1
1
sage: e2 = P(raw_element=2); e2
2^x
sage: e1 == e2
False
sage: P.le(e1, e2)
True
sage: P.le(e1, P(1)) and P.le(P(1), e2)
True
```

base

The base of this exponential growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('(ZZ_+)^x')
sage: P(42^x).base
42
```

class sage.rings.asymptotic.growth_group.**ExponentialGrowthGroup**(*base*, **args*, ***kwds*)
Bases: *sage.rings.asymptotic.growth_group.GenericGrowthGroup*

A growth group dealing with expressions involving a fixed variable/symbol as the exponent.

The elements *ExponentialGrowthElement* of this group represent exponential functions with bases from a fixed base ring; the group law is the multiplication.

INPUT:

- `base` – one of SageMath's parents, out of which the elements get their data (`raw_element`).

  As exponential expressions are represented by this group, the elements in `base` are the bases of these exponentials.

- `var` – an object.

  The string representation of `var` acts as an exponent of the elements represented by this group.

- `category` – (default: `None`) the category of the newly created growth group. It has to be a subcategory of `Join of Category of groups and Category of posets`. This is also the default category if `None` is specified.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import ExponentialGrowthGroup
sage: P = ExponentialGrowthGroup(QQ, 'x'); P
Growth Group QQ^x
```

See also:

GenericGrowthGroup

DivisionRings
alias of `sage.categories.division_rings.DivisionRings`

Element
alias of *ExponentialGrowthElement*

Groups
alias of `sage.categories.groups.Groups`

Magmas
alias of `sage.categories.magmas.Magmas`

Posets
alias of `sage.categories.posets.Posets`

Sets
alias of `sage.categories.sets_cat.Sets`

construction()
Return the construction of this growth group.

OUTPUT:

A pair whose first entry is an *exponential construction functor* and its second entry the base.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('(QQ_+)^x').construction()
(ExponentialGrowthGroup[x], Rational Field)
```

classmethod factory(*base*, *var*, *extend_by_non_growth_group=True*, *return_factors=False*, ***kwds*)
Create an exponential growth group.

This factory takes care of the splitting of the bases into their absolute values and arguments.

INPUT:

- `base`, `var`, keywords – use in the initialization of the exponential growth group; see *ExponentialGrowthGroup* for details.
- `extend_by_non_growth_group` – a boolean (default `True`). If set, then the growth group consists of two parts, one part dealing with the absolute values of the bases and one for their arguments.
- `return_factors` – a boolean (default: `False`). If set, then a tuple of the (cartesian) factors of this growth group is returned.

OUTPUT:

A growth group or tuple of growth groups.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import ExponentialGrowthGroup
sage: ExponentialGrowthGroup.factory(QQ, 'n')
Growth Group QQ^n * Signs^n
```

gens()

Return a tuple of all generators of this exponential growth group.

INPUT:

Nothing.

OUTPUT:

An empty tuple.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: E = GrowthGroup('(ZZ_+)^x')
sage: E.gens()
()
```

non_growth_group()

Return a non-growth group (with an argument group, e.g. roots of unity, as base) compatible with this exponential growth group.

OUTPUT:

A group group.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('(QQ_+)^x').non_growth_group()
Growth Group Signs^x
sage: GrowthGroup('(RR_+)^x').non_growth_group()
Growth Group Signs^x
sage: GrowthGroup('(RIF_+)^x').non_growth_group()
Growth Group Signs^x
sage: GrowthGroup('(RBF_+)^x').non_growth_group()
Growth Group Signs^x
sage: GrowthGroup('(CC_+)^x').non_growth_group()
Growth Group UU_RR^x
sage: GrowthGroup('(CIF_+)^x').non_growth_group()
Growth Group UU_RIF^x
sage: GrowthGroup('(CBF_+)^x').non_growth_group()
Growth Group UU_RBF^x
```

some_elements()

Return some elements of this exponential growth group.

See `TestSuite` for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: tuple(GrowthGroup('(QQ_+)^z').some_elements())
((1/2)^z, 2^z, 1, 42^z, (2/3)^z, (3/2)^z, ...)
```

class sage.rings.asymptotic.growth_group.**ExponentialGrowthGroupFunctor**(*var*)

Bases: *sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor*

A construction functor for *exponential growth groups*.

INPUT:

- var – a string or list of strings (or anything else *Variable* accepts).

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup,␣
↪ExponentialGrowthGroupFunctor
sage: GrowthGroup('(QQ_+)^z').construction()[0]
ExponentialGrowthGroup[z]
```

See also:

Asymptotic Ring, *AbstractGrowthGroupFunctor*, *MonomialGrowthGroupFunctor*, *sage.rings.asymptotic.asymptotic_ring.AsymptoticRingFunctor*, sage.categories.pushout.ConstructionFunctor.

class sage.rings.asymptotic.growth_group.**ExponentialNonGrowthElement**(*parent*, *raw_element*)

Bases: *sage.rings.asymptotic.growth_group.GenericNonGrowthElement*, *sage.rings.asymptotic.growth_group.ExponentialGrowthElement*

An element of *ExponentialNonGrowthGroup*.

class sage.rings.asymptotic.growth_group.**ExponentialNonGrowthGroup**(*base*, **args*, ***kwds*)

Bases: *sage.rings.asymptotic.growth_group.GenericNonGrowthGroup*, *sage.rings.asymptotic.growth_group.ExponentialGrowthGroup*

A growth group whose base is an argument group.

EXAMPLES:

```
sage: from sage.groups.misc_gps.argument_groups import RootsOfUnityGroup
sage: from sage.rings.asymptotic.growth_group import ExponentialNonGrowthGroup
sage: UU = ExponentialNonGrowthGroup(RootsOfUnityGroup(), 'n')
sage: UU(raw_element=-1)
(-1)^n
```

Element

alias of *ExponentialNonGrowthElement*

construction()

Return the construction of this growth group.

OUTPUT:

A pair whose first entry is an *ExponentialNonGrowthGroupFunctor* and its second entry the base.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('UU^x').construction()
(ExponentialNonGrowthGroup[x], Group of Roots of Unity)
```

class sage.rings.asymptotic.growth_group.**ExponentialNonGrowthGroupFunctor**(*var*)

Bases: *sage.rings.asymptotic.growth_group.ExponentialGrowthGroupFunctor*

A construction functor for *ExponentialNonGrowthGroup*.

class sage.rings.asymptotic.growth_group.**GenericGrowthElement**(*parent*, *raw_element*)

Bases: sage.structure.element.MultiplicativeGroupElement

A basic implementation of a generic growth element.

Growth elements form a group by multiplication, and (some of) the elements can be compared to each other, i.e., all elements form a poset.

INPUT:

- parent – a *GenericGrowthGroup*.
- raw_element – an element from the base of the parent.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import (GenericGrowthGroup,
....:                                                 GenericGrowthElement)
sage: G = GenericGrowthGroup(ZZ)
sage: g = GenericGrowthElement(G, 42); g
GenericGrowthElement(42)
sage: g.parent()
Growth Group Generic(ZZ)
sage: G(raw_element=42) == g
True
```

factors()

Return the atomic factors of this growth element. An atomic factor cannot be split further.

INPUT:

Nothing.

OUTPUT:

A tuple of growth elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ')
sage: G.an_element().factors()
(x,)
```

is_lt_one()

Return whether this element is less than 1.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ'); x = G(x)
sage: (x^42).is_lt_one()  # indirect doctest
False
sage: (x^(-42)).is_lt_one()  # indirect doctest
True
```

log(*base=None*)

Return the logarithm of this element.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ')
sage: x, = G.gens_monomial()
sage: log(x)  # indirect doctest
log(x)
sage: log(x^5)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(x^5) a factor 5 != 1 appeared,
which is not contained in Growth Group x^ZZ * log(x)^ZZ.
```

```
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ')
sage: x, = G.gens_monomial()
sage: el = x.rpow(2); el
2^x
sage: log(el)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(2^x) a factor log(2) != 1
appeared, which is not contained in Growth Group QQ^x * x^ZZ.
sage: log(el, base=2)  # indirect doctest
x
```

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: x = GenericGrowthGroup(ZZ).an_element()
sage: log(x)  # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: Cannot determine logarithmized factorization of
GenericGrowthElement(1) in abstract base class.
```

```
sage: x = GrowthGroup('x^ZZ').an_element()
sage: log(x)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(x) since log(x) is not in
Growth Group x^ZZ.
```

log_factor(*base=None*, *locals=None*)

Return the logarithm of the factorization of this element.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.
- `locals` – a dictionary which may contain the following keys and values:
  - `'log'` – value: a function. If not used, then the usual `log` is taken.

OUTPUT:

A tuple of pairs, where the first entry is a growth element and the second a multiplicative coefficient.

ALGORITHM:

This function factors the given element and calculates the logarithm of each of these factors.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ')
sage: x, y = G.gens_monomial()
sage: (x * y).log_factor()  # indirect doctest
((log(x), 1), (log(y), 1))
sage: (x^123).log_factor()  # indirect doctest
((log(x), 123),)
sage: (G('2^x') * x^2).log_factor(base=2)  # indirect doctest
((x, 1), (log(x), 2/log(2)))
```

```
sage: G(1).log_factor()
()
```

```
sage: log(x).log_factor()  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(log(x)) since log(log(x)) is
not in Growth Group QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ.
```

See also:

factors(), *log()*.

rpow(*base*)

Calculate the power of `base` to this element.

INPUT:

- `base` – an element.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ')
sage: x = G('x')
sage: x.rpow(2)  # indirect doctest
2^x
sage: x.rpow(1/2)  # indirect doctest
(1/2)^x
```

```
sage: x.rpow(0)  # indirect doctest
Traceback (most recent call last):
...
ValueError: 0 is not an allowed base for calculating the power to x.
sage: (x^2).rpow(2)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot construct 2^(x^2) in Growth Group QQ^x * x^ZZ
> *previous* TypeError: unsupported operand parent(s) for *:
'Growth Group QQ^x * x^ZZ' and 'Growth Group ZZ^(x^2)'
```

```
sage: G = GrowthGroup('QQ^(x*log(x)) * x^ZZ * log(x)^ZZ')
sage: x = G('x')
sage: (x * log(x)).rpow(2)  # indirect doctest
2^(x*log(x))
```

```
sage: n = GrowthGroup('(QQ_+)^n * n^QQ')('n')
sage: n.rpow(2)
2^n
sage: _.parent()
Growth Group QQ^n * n^QQ
```

```
sage: n = GrowthGroup('QQ^n * n^QQ')('n')
sage: n.rpow(-2)
2^n*(-1)^n
```

variable_names()

Return the names of the variables of this growth element.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('m^QQ')
sage: G('m^2').variable_names()
('m',)
sage: G('m^0').variable_names()
()
```

```
sage: G = GrowthGroup('QQ^m')
sage: G('2^m').variable_names()
('m',)
sage: G('1^m').variable_names()
()
```

class sage.rings.asymptotic.growth_group.**GenericGrowthGroup**(*base*, *var*, *category*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent, *sage.rings.asymptotic.misc.WithLocals*

A basic implementation for growth groups.

INPUT:

- `base` – one of SageMath's parents, out of which the elements get their data (`raw_element`).
- `category` – (default: `None`) the category of the newly created growth group. It has to be a subcategory of `Join of Category of groups and Category of posets`. This is also the default category if `None` is specified.
- `ignore_variables` – (default: `None`) a tuple (or other iterable) of strings. The specified names are not considered as variables.

Note: This class should be derived for concrete implementations.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: G = GenericGrowthGroup(ZZ); G
Growth Group Generic(ZZ)
```

See also:

MonomialGrowthGroup, *ExponentialGrowthGroup*

AdditiveMagmas
alias of sage.categories.additive_magmas.AdditiveMagmas

Element
alias of *GenericGrowthElement*

Magmas
alias of sage.categories.magmas.Magmas

Posets
alias of sage.categories.posets.Posets

Sets
alias of sage.categories.sets_cat.Sets

extended_by_non_growth_group()
Extend to a cartesian product of this growth group and a suitable non growth group.

OUTPUT:

A group group.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('(QQ_+)^x').extended_by_non_growth_group()
Growth Group QQ^x * Signs^x
sage: GrowthGroup('(RR_+)^x').extended_by_non_growth_group()
Growth Group RR^x * Signs^x
sage: GrowthGroup('(RIF_+)^x').extended_by_non_growth_group()
Growth Group RIF^x * Signs^x
sage: GrowthGroup('(RBF_+)^x').extended_by_non_growth_group()
Growth Group RBF^x * Signs^x
sage: GrowthGroup('(CC_+)^x').extended_by_non_growth_group()
Growth Group CC^x * UU_RR^x
sage: GrowthGroup('(CIF_+)^x').extended_by_non_growth_group()
Growth Group CIF^x * UU_RIF^x
sage: GrowthGroup('(CBF_+)^x').extended_by_non_growth_group()
Growth Group CBF^x * UU_RBF^x
```

gen(*n=0*)

Return the n-th generator (as a group) of this growth group.

INPUT:

- n – default: 0.

OUTPUT:

A *MonomialGrowthElement*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P.gen()
x
```

```
sage: P = GrowthGroup('(QQ_+)^x')
sage: P.gen()
Traceback (most recent call last):
...
IndexError: tuple index out of range
```

gens()

Return a tuple of all generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple whose entries are growth elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P.gens()
(x,)
```

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```
sage: GrowthGroup('log(x)^ZZ').gens()
(log(x),)
```

gens_monomial()

Return a tuple containing monomial generators of this growth group.

INPUT:

Nothing.

OUTPUT:

An empty tuple.

Note: A generator is called monomial generator if the variable of the underlying growth group is a valid identifier. For example, `x^ZZ` has `x` as a monomial generator, while `log(x)^ZZ` or `icecream(x)^ZZ` do not have monomial generators.

---

is_compatible(*other*)

Return whether this growth group is compatible with `other` meaning that both are of the same type and have the same variables, but maybe a different base.

INPUT:

- `other` – a growth group

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import ExponentialGrowthGroup,␣
↪ExponentialNonGrowthGroup
sage: from sage.groups.misc_gps.argument_groups import RootsOfUnityGroup
sage: EQ = ExponentialGrowthGroup(QQ, 'n')
sage: EZ = ExponentialGrowthGroup(ZZ, 'n')
sage: UU = ExponentialNonGrowthGroup(RootsOfUnityGroup(), 'n')
sage: for a in (EQ, EZ, UU):
....:     for b in (EQ, EZ, UU):
....:         print('{} is {}compatible with {}'.format(
....:             a, '' if a.is_compatible(b) else 'not ', b))
Growth Group QQ^n is compatible with Growth Group QQ^n
Growth Group QQ^n is compatible with Growth Group ZZ^n
Growth Group QQ^n is compatible with Growth Group UU^n
Growth Group ZZ^n is compatible with Growth Group QQ^n
Growth Group ZZ^n is compatible with Growth Group ZZ^n
Growth Group ZZ^n is compatible with Growth Group UU^n
Growth Group UU^n is not compatible with Growth Group QQ^n
Growth Group UU^n is not compatible with Growth Group ZZ^n
Growth Group UU^n is compatible with Growth Group UU^n
```

le(*left*, *right*)

Return whether the growth of `left` is at most (less than or equal to) the growth of `right`.

INPUT:

- `left` – an element.
- `right` – an element.

OUTPUT:

A boolean.

Note: This function uses the coercion model to find a common parent for the two operands.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ')
sage: x = G.gen()
sage: G.le(x, x^2)
True
sage: G.le(x^2, x)
False
sage: G.le(x^0, 1)
True
```

ngens()

Return the number of generators (as a group) of this growth group.

INPUT:

Nothing.

OUTPUT:

A Python integer.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P.ngens()
1
sage: GrowthGroup('log(x)^ZZ').ngens()
1
```

```
sage: P = GrowthGroup('(QQ_+)^x')
sage: P.ngens()
0
```

non_growth_group()

Return a non-growth group compatible with this growth group.

OUTPUT:

A group group.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: GenericGrowthGroup(ZZ, 'n').non_growth_group()
Traceback (most recent call last):
...
NotImplementedError: only implemented in concrete realizations
```

some_elements()

Return some elements of this growth group.

See `TestSuite` for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: tuple(GrowthGroup('z^ZZ').some_elements())
(1, z, z^(-1), z^2, z^(-2), z^3, z^(-3),
 z^4, z^(-4), z^5, z^(-5), ...)
sage: tuple(GrowthGroup('z^QQ').some_elements())
(z^(1/2), z^(-1/2), z^2, z^(-2),
 1, z, z^(-1), z^42,
 z^(2/3), z^(-2/3), z^(3/2), z^(-3/2),
 z^(4/5), z^(-4/5), z^(5/4), z^(-5/4), ...)
```

variable_names()

Return the names of the variables of this growth group.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: GenericGrowthGroup(ZZ).variable_names()
()
```

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ').variable_names()
('x',)
sage: GrowthGroup('log(x)^ZZ').variable_names()
('x',)
```

```
sage: GrowthGroup('(QQ_+)^x').variable_names()
('x',)
sage: GrowthGroup('(QQ_+)^(x*log(x))').variable_names()
('x',)
```

class sage.rings.asymptotic.growth_group.**GenericNonGrowthElement**(*parent*, *raw_element*)

Bases: *sage.rings.asymptotic.growth_group.GenericGrowthElement*

An element of *GenericNonGrowthGroup*.

class sage.rings.asymptotic.growth_group.**GenericNonGrowthGroup**(*base*, *var*, *category*)

Bases: *sage.rings.asymptotic.growth_group.GenericGrowthGroup*

A (abstract) growth group whose elements are all of the same growth 1.

See *ExponentialNonGrowthGroup* for a concrete realization.

sage.rings.asymptotic.growth_group.**GrowthGroup** =
<sage.rings.asymptotic.growth_group.GrowthGroupFactory object>
A factory for growth groups. This is an instance of *GrowthGroupFactory* whose documentation provides more details.

class sage.rings.asymptotic.growth_group.**GrowthGroupFactor**
Bases: tuple

base
Alias for field number 1

cls
Alias for field number 0

extend_by_non_growth_group
Alias for field number 3

var
Alias for field number 2

class sage.rings.asymptotic.growth_group.**GrowthGroupFactory**
Bases: sage.structure.factory.UniqueFactory

A factory creating asymptotic growth groups.

INPUT:

- specification – a string.
- keyword arguments are passed on to the growth group constructor. If the keyword ignore_variables is not specified, then ignore_variables=('e',) (to ignore e as a variable name) is used.

OUTPUT:

An asymptotic growth group.

Note: An instance of this factory is available as GrowthGroup.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ')
Growth Group x^ZZ
sage: GrowthGroup('log(x)^QQ')
Growth Group log(x)^QQ
```

This factory can also be used to construct Cartesian products of growth groups:

```
sage: GrowthGroup('x^ZZ * y^ZZ')
Growth Group x^ZZ * y^ZZ
sage: GrowthGroup('x^ZZ * log(x)^ZZ')
Growth Group x^ZZ * log(x)^ZZ
sage: GrowthGroup('x^ZZ * log(x)^ZZ * y^QQ')
Growth Group x^ZZ * log(x)^ZZ * y^QQ
sage: GrowthGroup('(QQ_+)^x * x^ZZ * y^QQ * (QQ_+)^z')
Growth Group QQ^x * x^ZZ * y^QQ * QQ^z
sage: GrowthGroup('QQ^x * x^ZZ * y^QQ * QQ^z')
Growth Group QQ^x * x^ZZ * Signs^x * y^QQ * QQ^z * Signs^z
```

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```
sage: GrowthGroup('exp(x)^ZZ * x^ZZ')
Growth Group exp(x)^ZZ * x^ZZ
sage: GrowthGroup('(e^x)^ZZ * x^ZZ')
Growth Group (e^x)^ZZ * x^ZZ
```

```
sage: GrowthGroup('QQ^n * n^ZZ')
Growth Group QQ^n * n^ZZ * Signs^n
sage: GrowthGroup('(QQ_+)^n * n^ZZ * UU^n')
Growth Group QQ^n * n^ZZ * UU^n
sage: GrowthGroup('(QQ_+)^n * n^ZZ')
Growth Group QQ^n * n^ZZ
```

```
sage: GrowthGroup('n^(ZZ)')
Growth Group n^ZZ
sage: GrowthGroup('n^(ZZ[I])')
Growth Group n^ZZ * n^(ZZ*I)
sage: GrowthGroup('n^(I*ZZ)')
Growth Group n^(ZZ*I)
sage: GrowthGroup('n^(ZZ*I)')
Growth Group n^(ZZ*I)
```

create_key_and_extra_args(*specification*, ***kwds*)
Given the arguments and keyword, create a key that uniquely determines this object.

create_object(*version*, *factors*, ***kwds*)
Create an object from the given arguments.

class sage.rings.asymptotic.growth_group.**MonomialGrowthElement**(*parent*, *raw_element*)
Bases: *sage.rings.asymptotic.growth_group.GenericGrowthElement*

An implementation of monomial growth elements.

INPUT:

- `parent` – a *MonomialGrowthGroup*.
- `raw_element` – an element from the base ring of the parent.

  This `raw_element` is the exponent of the created monomial growth element.

A monomial growth element represents a term of the type $\text{variable}^{\text{exponent}}$. The multiplication corresponds to the addition of the exponents.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import MonomialGrowthGroup
sage: P = MonomialGrowthGroup(ZZ, 'x')
sage: e1 = P(1); e1
1
sage: e2 = P(raw_element=2); e2
x^2
sage: e1 == e2
False
sage: P.le(e1, e2)
True
```

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```
sage: P.le(e1, P.gen()) and P.le(P.gen(), e2)
True
```

exponent
The exponent of this growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^ZZ')
sage: P(x^42).exponent
42
```

class sage.rings.asymptotic.growth_group.**MonomialGrowthGroup**(*base*, *var*, *category*)
Bases: *sage.rings.asymptotic.growth_group.GenericGrowthGroup*

A growth group dealing with powers of a fixed object/symbol.

The elements *MonomialGrowthElement* of this group represent powers of a fixed base; the group law is the multiplication, which corresponds to the addition of the exponents of the monomials.

INPUT:

- `base` – one of SageMath's parents, out of which the elements get their data (`raw_element`).

  As monomials are represented by this group, the elements in `base` are the exponents of these monomials.

- `var` – an object.

  The string representation of `var` acts as a base of the monomials represented by this group.

- `category` – (default: `None`) the category of the newly created growth group. It has to be a subcategory of `Join of Category of groups and Category of posets`. This is also the default category if `None` is specified.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import MonomialGrowthGroup
sage: P = MonomialGrowthGroup(ZZ, 'x'); P
Growth Group x^ZZ
sage: MonomialGrowthGroup(ZZ, log(SR.var('y')))
Growth Group log(y)^ZZ
```

See also:

GenericGrowthGroup

AdditiveMagmas
alias of sage.categories.additive_magmas.AdditiveMagmas

Element
alias of *MonomialGrowthElement*

Magmas
alias of sage.categories.magmas.Magmas

Posets
alias of sage.categories.posets.Posets

Sets
alias of sage.categories.sets_cat.Sets

construction()
Return the construction of this growth group.

OUTPUT:

A pair whose first entry is a *monomial construction functor* and its second entry the base.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ').construction()
(MonomialGrowthGroup[x], Integer Ring)
```

classmethod factory(*base*, *var*, *extend_by_non_growth_group=False*, *return_factors=False*, ***kwds*)
Create a monomial growth group.

INPUT:

- `base`, `var`, keywords – use in the initialization of the exponential growth group; see *MonomialGrowthGroup* for details.
- `extend_by_non_growth_group` – a boolean (default `False`). If set, then the growth group consists of two parts, one part dealing with the absolute values of the bases and one for their arguments.
- `return_factors` – a boolean (default: `False`). If set, then a tuple of the (cartesian) factors of this growth group is returned.

OUTPUT:

A growth group or tuple of growth groups.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import MonomialGrowthGroup
sage: from sage.groups.misc_gps.imaginary_groups import ImaginaryGroup
sage: MonomialGrowthGroup.factory(ZZ, 'n')
Growth Group n^ZZ
sage: MonomialGrowthGroup.factory(ImaginaryGroup(ZZ), 'n')
Growth Group n^(ZZ*I)
```

gens_logarithmic()
Return a tuple containing logarithmic generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple containing elements of this growth group.

Note: A generator is called logarithmic generator if the variable of the underlying growth group is the logarithm of a valid identifier. For example, `x^ZZ` has no logarithmic generator, while `log(x)^ZZ` has `log(x)` as logarithmic generator.

---

gens_monomial()
Return a tuple containing monomial generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple containing elements of this growth group.

Note: A generator is called monomial generator if the variable of the underlying growth group is a valid identifier. For example, `x^ZZ` has `x` as a monomial generator, while `log(x)^ZZ` or `icecream(x)^ZZ` do not have monomial generators.

---

non_growth_group()

Return a non-growth group (with an imaginary group as base) compatible with this monomial growth group.

OUTPUT:

A group group.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('n^ZZ').non_growth_group()
Growth Group n^(ZZ*I)
```

class sage.rings.asymptotic.growth_group.**MonomialGrowthGroupFunctor**(*var*)

Bases: *sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor*

A construction functor for *monomial growth groups*.

INPUT:

- `var` – a string or list of strings (or anything else *Variable* accepts).

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup,␣
↪MonomialGrowthGroupFunctor
sage: GrowthGroup('z^QQ').construction()[0]
MonomialGrowthGroup[z]
```

See also:

Asymptotic Ring, *AbstractGrowthGroupFunctor*, *ExponentialGrowthGroupFunctor*, *sage.rings.asymptotic.asymptotic_ring.AsymptoticRingFunctor*, sage.categories.pushout.ConstructionFunctor.

class sage.rings.asymptotic.growth_group.**MonomialNonGrowthElement**(*parent*, *raw_element*)

Bases: *sage.rings.asymptotic.growth_group.GenericNonGrowthElement*, *sage.rings.asymptotic.growth_group.MonomialGrowthElement*

An element of *MonomialNonGrowthGroup*.

class sage.rings.asymptotic.growth_group.**MonomialNonGrowthGroup**(*base*, *var*, *category*)

Bases: *sage.rings.asymptotic.growth_group.GenericNonGrowthGroup*, *sage.rings.asymptotic.growth_group.MonomialGrowthGroup*

A growth group whose base is an imaginary group.

EXAMPLES:

```
sage: from sage.groups.misc_gps.imaginary_groups import ImaginaryGroup
sage: from sage.rings.asymptotic.growth_group import MonomialNonGrowthGroup
sage: J = MonomialNonGrowthGroup(ImaginaryGroup(ZZ), 'n')
```

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```
sage: J.an_element()
n^I
```

Element
alias of *MonomialNonGrowthElement*

construction()
Return the construction of this growth group.

OUTPUT:

A pair whose first entry is an *MonomialNonGrowthGroupFunctor* and its second entry the base.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^(QQ*I)').construction()
(MonomialNonGrowthGroup[x], Imaginary Group over Rational Field)
```

class sage.rings.asymptotic.growth_group.**MonomialNonGrowthGroupFunctor**(*var*)
Bases: *sage.rings.asymptotic.growth_group.MonomialGrowthGroupFunctor*

A construction functor for *MonomialNonGrowthGroup*.

class sage.rings.asymptotic.growth_group.**PartialConversionElement**(*growth_group*, *raw_element*)
Bases: sage.structure.sage_object.SageObject

A not converted element of a growth group.

INPUT:

- growth_group – a group group
- raw_element – an object

A *PartialConversionElement* is an element growth_group(raw_element) which usually appears in conjunction with *PartialConversionValueError*. In this case, it was to possible to create that element, although the conversion went partially well in the sense that a $raw_element$ (e.g. an exponent for *MonomialGrowthElement* or a base for *ExponentialGrowthElement*) could be extracted.

Its main purpose is to carry data used during the creation of elements of *cartesian products of growth groups*.

is_compatible(*other*)
Wrapper to *GenericGrowthGroup.is_compatible()*.

split()
Split the contained raw_element according to the growth group's GrowthGroup._split_raw_element_().

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import ExponentialGrowthGroup,␣
↪PartialConversionValueError
sage: E = ExponentialGrowthGroup(ZZ, 'x')
sage: try:
....:     E((-2)^x)
....: except PartialConversionValueError as e:
....:     e.element.split()
(2^x, element with parameter -1 (<class 'int'>) in Growth Group ZZ^x)
```

exception sage.rings.asymptotic.growth_group.**PartialConversionValueError**(*element*, **args*, ***kwds*)

Bases: ValueError

A special ValueError which is raised when (partial) conversion fails.

INPUT:

- `element` – a *PartialConversionElement*

The remaining argument passed on to ValueError.

class sage.rings.asymptotic.growth_group.**Variable**(*var*, *repr=None*, *latex_name=None*, *ignore=None*)

Bases: sage.structure.unique_representation.CachedRepresentation, sage.structure.sage_object.SageObject

A class managing the variable of a growth group.

INPUT:

- `var` – an object whose representation string is used as the variable. It has to be a valid Python identifier. `var` can also be a tuple (or other iterable) of such objects.
- `repr` – (default: `None`) if specified, then this string will be displayed instead of `var`. Use this to get e.g. `log(x)^ZZ`: `var` is then used to specify the variable x.
- `latex_name` – (default: `None`) if specified, then this string will be used as LaTeX-representation of `var`.
- `ignore` – (default: `None`) a tuple (or other iterable) of strings which are not variables.

static extract_variable_names(*s*)

Determine the name of the variable for the given string.

INPUT:

- `s` – a string.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: Variable.extract_variable_names('')
()
sage: Variable.extract_variable_names('x')
('x',)
sage: Variable.extract_variable_names('exp(x)')
('x',)
sage: Variable.extract_variable_names('sin(cos(ln(x)))')
('x',)
```

```
sage: Variable.extract_variable_names('log(77w)')
('w',)
sage: Variable.extract_variable_names('log(x')
Traceback (most recent call last):
...
TypeError: Bad function call: log(x !!!
sage: Variable.extract_variable_names('x)')
Traceback (most recent call last):
```

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```
...
TypeError: Malformed expression: x) !!!
sage: Variable.extract_variable_names('log)x(')
Traceback (most recent call last):
...
TypeError: Malformed expression: log) !!! x(
sage: Variable.extract_variable_names('log(x)+y')
('x', 'y')
sage: Variable.extract_variable_names('icecream(summer)')
('summer',)
```

```
sage: Variable.extract_variable_names('a + b')
('a', 'b')
sage: Variable.extract_variable_names('a+b')
('a', 'b')
sage: Variable.extract_variable_names('a +b')
('a', 'b')
sage: Variable.extract_variable_names('+a')
('a',)
sage: Variable.extract_variable_names('a+')
Traceback (most recent call last):
...
TypeError: Malformed expression: a+ !!!
sage: Variable.extract_variable_names('b!')
('b',)
sage: Variable.extract_variable_names('-a')
('a',)
sage: Variable.extract_variable_names('a*b')
('a', 'b')
sage: Variable.extract_variable_names('2^q')
('q',)
sage: Variable.extract_variable_names('77')
()
```

```
sage: Variable.extract_variable_names('a + (b + c) + d')
('a', 'b', 'c', 'd')
```

is_monomial()

Return whether this is a monomial variable.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: Variable('x').is_monomial()
True
sage: Variable('log(x)').is_monomial()
False
```

variable_names()

Return the names of the variables.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import Variable
sage: Variable('x').variable_names()
('x',)
sage: Variable('log(x)').variable_names()
('x',)
```

4.4 Cartesian Products of Growth Groups

See *(Asymptotic) Growth Groups* for a description.

AUTHORS:

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- Clemens Heuberger (2016)

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4.4.1 Classes and Methods

class sage.rings.asymptotic.growth_group_cartesian.**CartesianProductFactory**

Bases: sage.structure.factory.UniqueFactory

Create various types of Cartesian products depending on its input.

INPUT:

- `growth_groups` – a tuple (or other iterable) of growth groups.
- `order` – (default: `None`) if specified, then this order is taken for comparing two Cartesian product elements. If `order` is `None` this is determined automatically.

Note: The Cartesian product of growth groups is again a growth group. In particular, the resulting structure is partially ordered.

The order on the product is determined as follows:

- Cartesian factors with respect to the same variable are ordered lexicographically. This causes `GrowthGroup('x^ZZ * log(x)^ZZ')` and `GrowthGroup('log(x)^ZZ * x^ZZ')` to produce two different growth groups.
- Factors over different variables are equipped with the product order (i.e. the comparison is componentwise).

Also, note that the sets of variables of the Cartesian factors have to be either equal or disjoint.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: A = GrowthGroup('x^ZZ'); A
Growth Group x^ZZ
sage: B = GrowthGroup('log(x)^ZZ'); B
Growth Group log(x)^ZZ
sage: C = cartesian_product([A, B]); C  # indirect doctest
Growth Group x^ZZ * log(x)^ZZ
sage: C._le_ == C.le_lex
True
sage: D = GrowthGroup('y^ZZ'); D
Growth Group y^ZZ
sage: E = cartesian_product([A, D]); E  # indirect doctest
Growth Group x^ZZ * y^ZZ
sage: E._le_ == E.le_product
True
sage: F = cartesian_product([C, D]); F  # indirect doctest
Growth Group x^ZZ * log(x)^ZZ * y^ZZ
sage: F._le_ == F.le_product
True
sage: cartesian_product([A, E]); G  # indirect doctest
Traceback (most recent call last):
...
ValueError: The growth groups (Growth Group x^ZZ, Growth Group x^ZZ * y^ZZ)
need to have pairwise disjoint or equal variables.
sage: cartesian_product([A, B, D])  # indirect doctest
Growth Group x^ZZ * log(x)^ZZ * y^ZZ
```

create_key_and_extra_args(*growth_groups*, *category*, ***kwds*)
Given the arguments and keywords, create a key that uniquely determines this object.

create_object(*version*, *args*, ***kwds*)
Create an object from the given arguments.

class sage.rings.asymptotic.growth_group_cartesian.**GenericProduct**(*sets*, *category*, ***kwds*)
Bases: sage.combinat.posets.cartesian_product.CartesianProductPoset, *sage.rings.asymptotic.growth_group.GenericGrowthGroup*

A Cartesian product of growth groups.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: P = GrowthGroup('x^QQ')
sage: L = GrowthGroup('log(x)^ZZ')
sage: C = cartesian_product([P, L], order='lex'); C  # indirect doctest
Growth Group x^QQ * log(x)^ZZ
sage: C.an_element()
x^(1/2)*log(x)
```

```
sage: Px = GrowthGroup('x^QQ')
sage: Lx = GrowthGroup('log(x)^ZZ')
```

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```
sage: Cx = cartesian_product([Px, Lx], order='lex')  # indirect doctest
sage: Py = GrowthGroup('y^QQ')
sage: C = cartesian_product([Cx, Py], order='product'); C  # indirect doctest
Growth Group x^QQ * log(x)^ZZ * y^QQ
sage: C.an_element()
x^(1/2)*log(x)*y^(1/2)
```

See also:

CartesianProduct, CartesianProductPoset.

class Element

Bases: sage.combinat.posets.cartesian_product.CartesianProductPoset.Element

exp()

The exponential of this element.

INPUT:

Nothing.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * log(log(x))^ZZ')
sage: x = G('x')
sage: exp(log(x))
x
sage: exp(log(log(x)))
log(x)
```

```
sage: exp(x)
Traceback (most recent call last):
...
ArithmeticError: Cannot construct e^x in
Growth Group x^ZZ * log(x)^ZZ * log(log(x))^ZZ
> *previous* TypeError: unsupported operand parent(s) for *:
'Growth Group x^ZZ * log(x)^ZZ * log(log(x))^ZZ' and
'Growth Group (e^x)^ZZ'
```

factors()

Return the atomic factors of this growth element. An atomic factor cannot be split further and is not the identity (1).

INPUT:

Nothing.

OUTPUT:

A tuple of growth elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * y^ZZ')
sage: x, y = G.gens_monomial()
sage: x.factors()
(x,)
sage: f = (x * y).factors(); f
(x, y)
sage: tuple(factor.parent() for factor in f)
(Growth Group x^ZZ, Growth Group y^ZZ)
sage: f = (x * log(x)).factors(); f
(x, log(x))
sage: tuple(factor.parent() for factor in f)
(Growth Group x^ZZ, Growth Group log(x)^ZZ)
```

```
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * log(log(x))^ZZ * y^QQ')
sage: x, y = G.gens_monomial()
sage: f = (x * log(x) * y).factors(); f
(x, log(x), y)
sage: tuple(factor.parent() for factor in f)
(Growth Group x^ZZ, Growth Group log(x)^ZZ, Growth Group y^QQ)
```

```
sage: G.one().factors()
()
```

is_lt_one()

Return whether this element is less than 1.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ'); x = G(x)
sage: (x^42).is_lt_one()  # indirect doctest
False
sage: (x^(-42)).is_lt_one()  # indirect doctest
True
```

log(*base=None*)

Return the logarithm of this element.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ')
sage: x, = G.gens_monomial()
sage: log(x)  # indirect doctest
log(x)
sage: log(x^5)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(x^5) a factor 5 != 1 appeared,
which is not contained in Growth Group x^ZZ * log(x)^ZZ.
```

```
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ')
sage: x, = G.gens_monomial()
sage: el = x.rpow(2); el
2^x
sage: log(el)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: When calculating log(2^x) a factor log(2) != 1
appeared, which is not contained in Growth Group QQ^x * x^ZZ.
sage: log(el, base=2)  # indirect doctest
x
```

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: x = GenericGrowthGroup(ZZ).an_element()
sage: log(x)  # indirect doctest
Traceback (most recent call last):
...
NotImplementedError: Cannot determine logarithmized factorization of
GenericGrowthElement(1) in abstract base class.
```

```
sage: x = GrowthGroup('x^ZZ').an_element()
sage: log(x)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(x) since log(x) is not in
Growth Group x^ZZ.
```

log_factor(*base=None*, *locals=None*)

Return the logarithm of the factorization of this element.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.
- `locals` – a dictionary which may contain the following keys and values:
  - `'log'` – value: a function. If not used, then the usual `log` is taken.

OUTPUT:

A tuple of pairs, where the first entry is a growth element and the second a multiplicative coefficient.

ALGORITHM:

This function factors the given element and calculates the logarithm of each of these factors.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ')
sage: x, y = G.gens_monomial()
sage: (x * y).log_factor()  # indirect doctest
((log(x), 1), (log(y), 1))
sage: (x^123).log_factor()  # indirect doctest
((log(x), 123),)
sage: (G('2^x') * x^2).log_factor(base=2)  # indirect doctest
((x, 1), (log(x), 2/log(2)))
```

```
sage: G(1).log_factor()
()
```

```
sage: log(x).log_factor()  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot build log(log(x)) since log(log(x)) is
not in Growth Group QQ^x * x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ.
```

See also:

factors(), *log()*.

rpow(*base*)

Calculate the power of `base` to this element.

INPUT:

- `base` – an element.

OUTPUT:

A growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('(QQ_+)^x * x^ZZ')
sage: x = G('x')
sage: x.rpow(2)  # indirect doctest
2^x
sage: x.rpow(1/2)  # indirect doctest
(1/2)^x
```

```
sage: x.rpow(0)  # indirect doctest
Traceback (most recent call last):
...
ValueError: 0 is not an allowed base for calculating the power to x.
sage: (x^2).rpow(2)  # indirect doctest
Traceback (most recent call last):
...
ArithmeticError: Cannot construct 2^(x^2) in Growth Group QQ^x * x^ZZ
> *previous* TypeError: unsupported operand parent(s) for *:
'Growth Group QQ^x * x^ZZ' and 'Growth Group ZZ^(x^2)'
```

```
sage: G = GrowthGroup('QQ^(x*log(x)) * x^ZZ * log(x)^ZZ')
sage: x = G('x')
sage: (x * log(x)).rpow(2)  # indirect doctest
2^(x*log(x))
```

```
sage: n = GrowthGroup('(QQ_+)^n * n^QQ')('n')
sage: n.rpow(2)
2^n
sage: _.parent()
Growth Group QQ^n * n^QQ
```

```
sage: n = GrowthGroup('QQ^n * n^QQ')('n')
sage: n.rpow(-2)
2^n*(-1)^n
```

variable_names()

Return the names of the variables of this growth element.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('QQ^m * m^QQ * log(n)^ZZ')
sage: G('2^m * m^4 * log(n)').variable_names()
('m', 'n')
sage: G('2^m * m^4').variable_names()
('m',)
sage: G('log(n)').variable_names()
('n',)
sage: G('m^3').variable_names()
('m',)
sage: G('m^0').variable_names()
()
```

cartesian_injection(*factor*, *element*)

Inject the given element into this Cartesian product at the given factor.

INPUT:

- `factor` – a growth group (a factor of this Cartesian product).
- `element` – an element of `factor`.

OUTPUT:

An element of this Cartesian product.

gens_monomial()

Return a tuple containing monomial generators of this growth group.

INPUT:

Nothing.

OUTPUT:

A tuple containing elements of this growth group.

Note: This method calls the `gens_monomial()` method on the individual factors of this Cartesian product and concatenates the respective outputs.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * log(x)^ZZ * y^QQ * log(z)^ZZ')
sage: G.gens_monomial()
(x, y)
```

some_elements()

Return some elements of this Cartesian product of growth groups.

See `TestSuite` for a typical use case.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from itertools import islice
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('(QQ_+)^y * x^QQ * log(x)^ZZ')
sage: tuple(islice(G.some_elements(), 10r))
(x^(1/2)*(1/2)^y,
 x^(-1/2)*log(x)*2^y,
 x^2*log(x)^(-1),
 x^(-2)*log(x)^2*42^y,
 log(x)^(-2)*(2/3)^y,
 x*log(x)^3*(3/2)^y,
 x^(-1)*log(x)^(-3)*(4/5)^y,
 x^42*log(x)^4*(5/4)^y,
 x^(2/3)*log(x)^(-4)*(6/7)^y,
 x^(-2/3)*log(x)^5*(7/6)^y)
```

variable_names()

Return the names of the variables.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: GrowthGroup('x^ZZ * log(x)^ZZ * y^QQ * log(z)^ZZ').variable_names()
('x', 'y', 'z')
```

class sage.rings.asymptotic.growth_group_cartesian.**MultivariateProduct**(*sets*, *category*, ***kwargs*)

Bases: *sage.rings.asymptotic.growth_group_cartesian.GenericProduct*

A Cartesian product of growth groups with pairwise disjoint (or equal) variable sets.

Note: A multivariate product of growth groups is ordered by means of the product order, i.e. componentwise. This is motivated by the assumption that different variables are considered to be independent (e.g. `x^ZZ * y^ZZ`).

See also:

UnivariateProduct, *GenericProduct*.

class sage.rings.asymptotic.growth_group_cartesian.**UnivariateProduct**(*sets*, *category*, ***kwargs*)
Bases: *sage.rings.asymptotic.growth_group_cartesian.GenericProduct*

A Cartesian product of growth groups with the same variables.

Note: A univariate product of growth groups is ordered lexicographically. This is motivated by the assumption that univariate growth groups can be ordered in a chain with respect to the growth they model (e.g. `x^ZZ * log(x)^ZZ`: polynomial growth dominates logarithmic growth).

See also:

MultivariateProduct, *GenericProduct*.

4.5 (Asymptotic) Term Monoids

This module implements asymptotic term monoids. The elements of these monoids are used behind the scenes when performing calculations in an *asymptotic ring*.

The monoids build upon the (asymptotic) growth groups. While growth elements only model the growth of a function as it tends towards infinity (or tends towards another fixed point; see *(Asymptotic) Growth Groups* for more details), an asymptotic term additionally specifies its "type" and performs the actual arithmetic operations (multiplication and partial addition/absorption of terms).

Besides an abstract base term *GenericTerm*, this module implements the following types of terms:

- *OTerm* – O-terms at infinity, see Wikipedia article Big_O_notation.
- *TermWithCoefficient* – abstract base class for asymptotic terms with coefficients.
- *ExactTerm* – this class represents a growth element multiplied with some non-zero coefficient from a coefficient ring.

A characteristic property of asymptotic terms is that some terms are able to "absorb" other terms (see *absorb()*). For instance, $O(x^2)$ is able to absorb $O(x)$ (with result $O(x^2)$), and $3 \cdot x^5$ is able to absorb $-2 \cdot x^5$ (with result x^5). Essentially, absorption can be interpreted as the addition of "compatible" terms (partial addition).

4.5.1 Absorption of Asymptotic Terms

A characteristic property of asymptotic terms is that some terms are able to "absorb" other terms. This is realized with the method *absorb()*.

For instance, $O(x^2)$ is able to absorb $O(x)$ (with result $O(x^2)$). This is because the functions bounded by linear growth are bounded by quadratic growth as well. Another example would be that $3x^5$ is able to absorb $-2x^5$ (with result x^5), which simply corresponds to addition.

Essentially, absorption can be interpreted as the addition of "compatible" terms (partial addition).

We want to show step by step which terms can be absorbed by which other terms. We start by defining the necessary term monoids and some terms:

```
sage: from sage.rings.asymptotic.term_monoid import OTermMonoid, ExactTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: OT = OTermMonoid(TermMonoid, growth_group=G, coefficient_ring=QQ)
sage: ET = ExactTermMonoid(TermMonoid, growth_group=G, coefficient_ring=QQ)
sage: ot1 = OT(x); ot2 = OT(x^2)
sage: et1 = ET(x^2, coefficient=2)
```

- Because of the definition of O-terms (see Wikipedia article Big_O_notation), *OTerm* are able to absorb all other asymptotic terms with weaker or equal growth. In our implementation, this means that *OTerm* is able to absorb other *OTerm*, as well as *ExactTerm*, as long as the growth of the other term is less than or equal to the growth of this element:

  ```
  sage: ot1, ot2
  (O(x), O(x^2))
  sage: ot1.can_absorb(ot2), ot2.can_absorb(ot1)
  (False, True)
  sage: et1
  2*x^2
  sage: ot1.can_absorb(et1)
  False
  sage: ot2.can_absorb(et1)
  True
  ```

  The result of this absorption always is the dominant (absorbing) *OTerm*:

  ```
  sage: ot1.absorb(ot1)
  O(x)
  sage: ot2.absorb(ot1)
  O(x^2)
  sage: ot2.absorb(et1)
  O(x^2)
  ```

  These examples correspond to $O(x) + O(x) = O(x)$, $O(x^2) + O(x) = O(x^2)$, and $O(x^2) + 2x^2 = O(x^2)$.

- *ExactTerm* can only absorb another *ExactTerm* if the growth coincides with the growth of this element:

  ```
  sage: et1.can_absorb(ET(x^2, coefficient=5))
  True
  sage: any(et1.can_absorb(t) for t in [ot1, ot2])
  False
  ```

  As mentioned above, absorption directly corresponds to addition in this case:

  ```
  sage: et1.absorb(ET(x^2, coefficient=5))
  7*x^2
  ```

  When adding two exact terms, they might cancel out. For technical reasons, `None` is returned in this case:

```
sage: ET(x^2, coefficient=5).can_absorb(ET(x^2, coefficient=-5))
True
sage: ET(x^2, coefficient=5).absorb(ET(x^2, coefficient=-5)) is None
True
```

- The abstract base terms *GenericTerm* and *TermWithCoefficient* can neither absorb any other term, nor be absorbed by any other term.

If `absorb` is called on a term that cannot be absorbed, an ArithmeticError is raised:

```
sage: ot1.absorb(ot2)
Traceback (most recent call last):
...
ArithmeticError: O(x) cannot absorb O(x^2)
```

This would only work the other way around:

```
sage: ot2.absorb(ot1)
O(x^2)
```

4.5.2 Comparison

The comparison of asymptotic terms with $\leq$ is implemented as follows:

- When comparing `t1 <= t2`, the coercion framework first tries to find a common parent for both terms. If this fails, `False` is returned.
- In case the coerced terms do not have a coefficient in their common parent (e.g. *OTerm*), the growth of the two terms is compared.
- Otherwise, if the coerced terms have a coefficient (e.g. *ExactTerm*), we compare whether `t1` has a growth that is strictly weaker than the growth of `t2`. If so, we return `True`. If the terms have equal growth, then we return `True` if and only if the coefficients coincide as well.

  In all other cases, we return `False`.

Long story short: we consider terms with different coefficients that have equal growth to be incomparable.

4.5.3 Various

Warning: The code for *B-Terms* is experimental, so a warning is thrown when a *BTerm* is created for the first time in a session (see `sage.misc.superseded.experimental`).

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('B', growth_group=GrowthGroup('x^ZZ'), coefficient_ring=QQ)
doctest:warning
...
FutureWarning: This class/method/function is marked as experimental.
It, its functionality or its interface might change without a formal deprecation.
See https://trac.sagemath.org/31922 for details.
```

Todo:

- Implementation of more term types (e.g. Ω terms, o terms, Θ terms).

AUTHORS:

- Benjamin Hackl (2015)
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- Clemens Heuberger (2016)
- Thomas Hagelmayer (2021)

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4.5.4 Classes and Methods

class sage.rings.asymptotic.term_monoid.**BTerm**(*parent*, *growth*, *valid_from*, ***kwds*)

Bases: *sage.rings.asymptotic.term_monoid.TermWithCoefficient*

Class for asymptotic B-terms.

A B-term represents all functions which (in absolute value) are bounded by the given `growth` and `coefficient` for the parameters given by `valid_from`. For example, we have terms that represent functions

- bounded by $5|x|^2$ for $|x| \geq 3$,
- bounded by $42|x|^3$ for $|x| \geq 15$ and $|y| \geq 15$, or
- bounded by $42|x|^3|y|^2$ for $|x| \geq 10$ and $|y| \geq 20$ (see below for the actual examples).

INPUT:

- `parent` – the parent of the asymptotic term
- `growth` – an asymptotic growth element of the parent's growth group
- `coefficient` – an element of the parent's coefficient ring
- `valid_from` – dictionary mapping variable names to lower bounds for the corresponding variable. The bound implied by this term is valid when all variables are at least their corresponding lower bound. If a number is passed to `valid_from`, then the lower bounds for all variables of the asymptotic expansion are set to this number

EXAMPLES:

We revisit the examples from the introduction:

```
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('x^ZZ * y^ZZ')
sage: T = TermMonoid('B', growth_group=G, coefficient_ring=ZZ)
sage: x, y = G('x'), G('y')
```

This is a term bounded by $5|x|^2$ for $|x| \geq 3$:

```
sage: T(x^2, coefficient=5, valid_from={'x': 3})
B(5*x^2, x >= 3)
```

This is a term bounded by $42|x|^3$ for $|x| \geq 15$ and $|y| \geq 15$:

```
sage: T(x^3, coefficient=42, valid_from={'x': 15, 'y': 15})
B(42*x^3, x >= 15, y >= 15)
```

This is a term bounded by $42|x|^3|y|^2$ for $|x| \geq 10$ and $|y| \geq 20$:

```
sage: T(x^3*y^2, coefficient=42, valid_from={'x': 10, 'y': 20})
B(42*x^3*y^2, x >= 10, y >= 20)
```

can_absorb(*other*)

Check whether this B-term can absorb `other`.

INPUT:

- `other` – an asymptotic term

OUTPUT:

A boolean

Note: A *BTerm* can absorb another *BTerm* with weaker or equal growth.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: BT = TermMonoid('B', GrowthGroup('x^ZZ'), QQ)
sage: t1 = BT(x^3, coefficient=3, valid_from={'x': 20})
sage: t2 = BT(x^2, coefficient=1, valid_from={'x': 10})
sage: t3 = BT(x^3, coefficient=10, valid_from={'x': 10})
sage: t1.can_absorb(t2)
True
sage: t2.can_absorb(t1)
False
sage: t1.can_absorb(t3)
True
sage: t3.can_absorb(t1)
True
sage: ET = TermMonoid('exact', GrowthGroup('x^ZZ'), QQ)
sage: t4 = ET(x^3, coefficient=5)
sage: t1.can_absorb(t4)
True
```

construction()

Return a construction of this term.

INPUT:

Nothing.

OUTPUT:

A pair `(cls, kwds)` such that `cls(**kwds)` equals this term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermMonoidFactory
sage: TermMonoid = TermMonoidFactory('__main__.TermMonoid')

sage: T = TermMonoid('B', GrowthGroup('x^ZZ'), QQ)
sage: a = T.an_element(); a
B(1/2*x, x >= 42)
sage: cls, kwds = a.construction(); cls, kwds
(<class 'sage.rings.asymptotic.term_monoid.BTermMonoid_with_category.element_
↪class'>,
 {'coefficient': 1/2,
  'growth': x,
  'parent': B-Term Monoid x^ZZ with coefficients in Rational Field,
  'valid_from': {'x': 42}})
sage: cls(**kwds) == a
True
```

See also:

GenericTerm.construction(), *TermWithCoefficient.construction()*, *GenericTermMonoid.from_construction()*

class sage.rings.asymptotic.term_monoid.**BTermMonoid**(*term_monoid_factory*, *growth_group*, *coefficient_ring*, *category*)

Bases: *sage.rings.asymptotic.term_monoid.TermWithCoefficientMonoid*

Parent for asymptotic B-terms.

INPUT:

- `growth_group` – a growth group
- `coefficient_ring` – the ring which contains the coefficients of the elements
- `category` – The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of `Join of Category of monoids and Category of posets`. This is also the default category if `None` is specified

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import MonomialGrowthGroup
sage: from sage.rings.asymptotic.term_monoid import BTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G = MonomialGrowthGroup(ZZ, 'x')
sage: BT = TermMonoid('B', G, QQ)
sage: BT
B-Term Monoid x^ZZ with coefficients in Rational Field
sage: BT is BTermMonoid(TermMonoid, G, QQ)
True
```

Element
alias of *BTerm*

some_elements()
Return some elements of this B-term monoid.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from itertools import islice
sage: from sage.rings.asymptotic.term_monoid import TermMonoidFactory
sage: TermMonoid = TermMonoidFactory('__main__.TermMonoid')
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('z^QQ')
sage: T = TermMonoid('B', G, ZZ)
sage: tuple(islice(T.some_elements(), int(10)))
(B(z^(1/2), z >= 0),
 B(z^(-1/2), z >= 1),
 B(z^(1/2), z >= 3),
 B(z^2, z >= 42),
 B(z^(-1/2), z >= 0),
 B(2*z^(1/2), z >= 1),
 B(z^(-2), z >= 3),
 B(z^2, z >= 42),
 B(2*z^(-1/2), z >= 0),
 B(2*z^(1/2), z >= 1))
```

sage.rings.asymptotic.term_monoid.**DefaultTermMonoidFactory = Term Monoid Factory 'sage.rings.asymptotic.term_monoid.DefaultTermMonoidFactory'**
A factory for asymptotic term monoids. This is an instance of *TermMonoidFactory* whose documentation provides more details.

class sage.rings.asymptotic.term_monoid.**ExactTerm**(*parent*, *growth*, *coefficient*)
Bases: *sage.rings.asymptotic.term_monoid.TermWithCoefficient*

Class for asymptotic exact terms. These terms primarily consist of an asymptotic growth element as well as a coefficient.

INPUT:

- `parent` – the parent of the asymptotic term.
- `growth` – an asymptotic growth element from `parent.growth_group`.
- `coefficient` – an element from `parent.coefficient_ring`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: from sage.rings.asymptotic.term_monoid import ExactTermMonoid
```

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```
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: ET = ExactTermMonoid(TermMonoid, G, QQ)
```

Asymptotic exact terms may be multiplied (with the usual rules applying):

```
sage: ET(x^2, coefficient=3) * ET(x, coefficient=-1)
-3*x^3
sage: ET(x^0, coefficient=4) * ET(x^5, coefficient=2)
8*x^5
```

They may also be multiplied with O-terms:

```
sage: OT = TermMonoid('O', G, QQ)
sage: ET(x^2, coefficient=42) * OT(x)
O(x^3)
```

Absorption for asymptotic exact terms relates to addition:

```
sage: ET(x^2, coefficient=5).can_absorb(ET(x^5, coefficient=12))
False
sage: ET(x^2, coefficient=5).can_absorb(ET(x^2, coefficient=1))
True
sage: ET(x^2, coefficient=5).absorb(ET(x^2, coefficient=1))
6*x^2
```

Note that, as for technical reasons, 0 is not allowed as a coefficient for an asymptotic term with coefficient. Instead `None` is returned if two asymptotic exact terms cancel out each other during absorption:

```
sage: ET(x^2, coefficient=42).can_absorb(ET(x^2, coefficient=-42))
True
sage: ET(x^2, coefficient=42).absorb(ET(x^2, coefficient=-42)) is None
True
```

Exact terms can also be created by converting monomials with coefficient from the symbolic ring, or a suitable polynomial or power series ring:

```
sage: x = var('x'); x.parent()
Symbolic Ring
sage: ET(5*x^2)
5*x^2
```

can_absorb(*other*)

Check whether this exact term can absorb `other`.

INPUT:

- `other` – an asymptotic term.

OUTPUT:

A boolean.

Note: For *ExactTerm*, absorption corresponds to addition. This means that an exact term can absorb only other exact terms with the same growth.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: ET = TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ)
sage: t1 = ET(x^21, coefficient=1); t2 = ET(x^21, coefficient=2); t3 = ET(x^42,␣
↪coefficient=1)
sage: t1.can_absorb(t2)
True
sage: t2.can_absorb(t1)
True
sage: t1.can_absorb(t3) or t3.can_absorb(t1)
False
```

is_constant()

Return whether this term is an (exact) constant.

INPUT:

Nothing.

OUTPUT:

A boolean.

Note: Only *ExactTerm* with constant growth (1) are constant.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T('x * log(x)').is_constant()
False
sage: T('3*x').is_constant()
False
sage: T(1/2).is_constant()
True
sage: T(42).is_constant()
True
```

is_exact()

Return whether this term is an exact term.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
```

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```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T('x * log(x)').is_exact()
True
sage: T('3 * x^2').is_exact()
True
```

is_little_o_of_one()

Return whether this exact term is of order $o(1)$.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ'), QQ)
sage: T(x).is_little_o_of_one()
False
sage: T(1).is_little_o_of_one()
False
sage: T(x^(-1)).is_little_o_of_one()
True
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * y^ZZ'), QQ)
sage: T('x * y^(-1)').is_little_o_of_one()
False
sage: T('x^(-1) * y').is_little_o_of_one()
False
sage: T('x^(-2) * y^(-3)').is_little_o_of_one()
True
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^QQ * log(x)^QQ'), QQ)
sage: T('x * log(x)^2').is_little_o_of_one()
False
sage: T('x^2 * log(x)^(-1234)').is_little_o_of_one()
False
sage: T('x^(-1) * log(x)^4242').is_little_o_of_one()
True
sage: T('x^(-1/100) * log(x)^(1000/7)').is_little_o_of_one()
True
```

log_term(*base=None*, *locals=None*)

Determine the logarithm of this exact term.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.
- `locals` – a dictionary which may contain the following keys and values:

- 'log' – value: a function. If not used, then the usual log is taken.

OUTPUT:

A tuple of terms.

Note: This method returns a tuple with the summands that come from applying the rule $\log(x \cdot y) = \log(x) + \log(y)$.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ'), SR)
sage: T(3*x^2).log_term()
(log(3), 2*log(x))
sage: T(x^1234).log_term()
(1234*log(x),)
sage: T(49*x^7).log_term(base=7)
(2, 7/log(7)*log(x))
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ
↪'), SR)
sage: T('x * y').log_term()
(log(x), log(y))
sage: T('4 * x * y').log_term(base=2)
(2, 1/log(2)*log(x), 1/log(2)*log(y))
```

See also:

OTerm.log_term().

rpow(*base*)

Return the power of base to this exact term.

INPUT:

- base – an element or 'e'.

OUTPUT:

A term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('exact', GrowthGroup('QQ^x * x^ZZ * log(x)^ZZ'), QQ)
sage: T('x').rpow(2)
2^x
sage: T('log(x)').rpow('e')
x
sage: T('42*log(x)').rpow('e')
x^42
```

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```
sage: T('3*x').rpow(2)
8^x
```

```
sage: T('3*x^2').rpow(2)
Traceback (most recent call last):
...
ArithmeticError: Cannot construct 2^(x^2) in
Growth Group QQ^x * x^ZZ * log(x)^ZZ * Signs^x
> *previous* TypeError: unsupported operand parent(s) for *:
'Growth Group QQ^x * x^ZZ * log(x)^ZZ * Signs^x' and
'Growth Group ZZ^(x^2)'
```

```
sage: T = TermMonoid('exact', GrowthGroup('(QQ_+)^n * n^QQ'), SR)
sage: n = T('n')
sage: n.rpow(2)
2^n
sage: _.parent()
Exact Term Monoid QQ^n * n^QQ with coefficients in Symbolic Ring
```

class sage.rings.asymptotic.term_monoid.**ExactTermMonoid**(*term_monoid_factory*, *growth_group*, *coefficient_ring*, *category*)

Bases: *sage.rings.asymptotic.term_monoid.TermWithCoefficientMonoid*

Parent for asymptotic exact terms, implemented in *ExactTerm*.

INPUT:

- `growth_group` – a growth group.
- `category` – The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of `Join of Category of monoids and Category of posets`. This is also the default category if `None` is specified.
- `coefficient_ring` – the ring which contains the coefficients of the elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import ExactTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G_ZZ = GrowthGroup('x^ZZ'); x_ZZ = G_ZZ.gen()
sage: G_QQ = GrowthGroup('x^QQ'); x_QQ = G_QQ.gen()
sage: ET_ZZ = ExactTermMonoid(TermMonoid, G_ZZ, ZZ); ET_ZZ
Exact Term Monoid x^ZZ with coefficients in Integer Ring
sage: ET_QQ = ExactTermMonoid(TermMonoid, G_QQ, QQ); ET_QQ
Exact Term Monoid x^QQ with coefficients in Rational Field
sage: ET_QQ.coerce_map_from(ET_ZZ)
Coercion map:
  From: Exact Term Monoid x^ZZ with coefficients in Integer Ring
  To:   Exact Term Monoid x^QQ with coefficients in Rational Field
```

Exact term monoids can also be created using the *term factory*:

```
sage: TermMonoid('exact', G_ZZ, ZZ) is ET_ZZ
True
sage: TermMonoid('exact', GrowthGroup('x^ZZ'), QQ)
Exact Term Monoid x^ZZ with coefficients in Rational Field
```

Element
alias of *ExactTerm*

class sage.rings.asymptotic.term_monoid.**GenericTerm**(*parent*, *growth*)
Bases: sage.structure.element.MultiplicativeGroupElement

Base class for asymptotic terms. Mainly the structure and several properties of asymptotic terms are handled here.

INPUT:

- `parent` – the parent of the asymptotic term.
- `growth` – an asymptotic growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: T = GenericTermMonoid(TermMonoid, G, QQ)
sage: t1 = T(x); t2 = T(x^2); (t1, t2)
(Generic Term with growth x, Generic Term with growth x^2)
sage: t1 * t2
Generic Term with growth x^3
sage: t1.can_absorb(t2)
False
sage: t1.absorb(t2)
Traceback (most recent call last):
...
ArithmeticError: Generic Term with growth x cannot absorb Generic Term with growth␣
↪x^2
sage: t1.can_absorb(t1)
False
```

absorb(*other*, *check=True*)
Absorb the asymptotic term `other` and return the resulting asymptotic term.

INPUT:

- `other` – an asymptotic term.
- `check` – a boolean. If `check` is `True` (default), then `can_absorb` is called before absorption.

OUTPUT:

An asymptotic term or `None` if a cancellation occurs. If no absorption can be performed, an ArithmeticError is raised.

Note: Setting `check` to `False` is meant to be used in cases where the respective comparison is done externally (in order to avoid duplicate checking).

For a more detailed explanation of the *absorption* of asymptotic terms see the *module description*.

EXAMPLES:

We want to demonstrate in which cases an asymptotic term is able to absorb another term, as well as explain the output of this operation. We start by defining some parents and elements:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G_QQ = GrowthGroup('x^QQ'); x = G_QQ.gen()
sage: OT = TermMonoid('O', G_QQ, coefficient_ring=QQ)
sage: ET = TermMonoid('exact', G_QQ, coefficient_ring=QQ)
sage: ot1 = OT(x); ot2 = OT(x^2)
sage: et1 = ET(x, coefficient=100); et2 = ET(x^2, coefficient=2)
sage: et3 = ET(x^2, coefficient=1); et4 = ET(x^2, coefficient=-2)
```

O-Terms are able to absorb other O-terms and exact terms with weaker or equal growth.

```
sage: ot1.absorb(ot1)
O(x)
sage: ot1.absorb(et1)
O(x)
sage: ot1.absorb(et1) is ot1
True
```

ExactTerm is able to absorb another *ExactTerm* if the terms have the same growth. In this case, *absorption* is nothing else than an addition of the respective coefficients:

```
sage: et2.absorb(et3)
3*x^2
sage: et3.absorb(et2)
3*x^2
sage: et3.absorb(et4)
-x^2
```

Note that, for technical reasons, the coefficient 0 is not allowed, and thus `None` is returned if two exact terms cancel each other out:

```
sage: et2.absorb(et4)
sage: et4.absorb(et2) is None
True
```

can_absorb(*other*)

Check whether this asymptotic term is able to absorb the asymptotic term `other`.

INPUT:

- `other` – an asymptotic term.

OUTPUT:

A boolean.

Note: A *GenericTerm* cannot absorb any other term.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GenericGrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G = GenericGrowthGroup(ZZ)
sage: T = GenericTermMonoid(TermMonoid, G, QQ)
sage: g1 = G(raw_element=21); g2 = G(raw_element=42)
sage: t1 = T(g1); t2 = T(g2)
sage: t1.can_absorb(t2)  # indirect doctest
False
sage: t2.can_absorb(t1)  # indirect doctest
False
```

construction()

Return a construction of this term.

INPUT:

Nothing.

OUTPUT:

A pair `(cls, kwds)` such that `cls(**kwds)` equals this term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: a = T.an_element(); a
O(x)
sage: cls, kwds = a.construction(); cls, kwds
(<class 'sage.rings.asymptotic.term_monoid.OTermMonoid_with_category.element_
↪class'>,
{'growth': x,
 'parent': O-Term Monoid x^ZZ with implicit coefficients in Rational Field})
sage: cls(**kwds) == a
True
```

See also:

TermWithCoefficient.construction(), *GenericTermMonoid.from_construction()*

is_constant()

Return whether this term is an (exact) constant.

INPUT:

Nothing.

OUTPUT:

A boolean.

Note: Only *ExactTerm* with constant growth (1) are constant.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = GenericTermMonoid(TermMonoid, GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: t = T.an_element(); t
Generic Term with growth x*log(x)
sage: t.is_constant()
False
```

```
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: T('x').is_constant()
False
sage: T(1).is_constant()
False
```

is_exact()

Return whether this term is an exact term.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = GenericTermMonoid(TermMonoid, GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T.an_element().is_exact()
False
```

is_little_o_of_one()

Return whether this generic term is of order $o(1)$.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import (GenericTermMonoid,
....:                                                TermWithCoefficientMonoid)
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
```

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```
sage: T = GenericTermMonoid(TermMonoid, GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().is_little_o_of_one()
Traceback (most recent call last):
...
NotImplementedError: Cannot check whether Generic Term with growth x is o(1)
in the abstract base class
GenericTerm Monoid x^ZZ with (implicit) coefficients in Rational Field.
sage: T = TermWithCoefficientMonoid(TermMonoid, GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().is_little_o_of_one()
Traceback (most recent call last):
...
NotImplementedError: Cannot check whether Term with coefficient 1/2 and growth x
is o(1) in the abstract base class
TermWithCoefficient Monoid x^ZZ with coefficients in Rational Field.
```

log_term(*base=None*, *locals=None*)

Determine the logarithm of this term.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.
- `locals` – a dictionary which may contain the following keys and values:
  - `'log'` – value: a function. If not used, then the usual `log` is taken.

OUTPUT:

A tuple of terms.

Note: This abstract method raises a NotImplementedError. See *ExactTerm* and *OTerm* for a concrete implementation.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: T = GenericTermMonoid(TermMonoid, GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().log_term()
Traceback (most recent call last):
...
NotImplementedError: This method is not implemented in
this abstract base class.
```

```
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: T = TermWithCoefficientMonoid(TermMonoid, GrowthGroup('x^ZZ'), QQ)
sage: T.an_element().log_term()
Traceback (most recent call last):
...
```

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```
NotImplementedError: This method is not implemented in
this abstract base class.
```

See also:

ExactTerm.log_term(), *OTerm.log_term()*.

rpow(*base*)

Return the power of `base` to this generic term.

INPUT:

- `base` – an element or `'e'`.

OUTPUT:

A term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: T = GenericTermMonoid(TermMonoid, GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T.an_element().rpow('e')
Traceback (most recent call last):
...
NotImplementedError: Cannot take e to the exponent
Generic Term with growth x*log(x) in the abstract base class
GenericTerm Monoid x^ZZ * log(x)^ZZ with (implicit) coefficients in Rational␣
↪Field.
```

variable_names()

Return the names of the variables of this term.

OUTPUT:

A tuple of strings.

EXAMPLES:

```
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('exact', 'QQ^m * m^QQ * log(n)^ZZ', QQ)
sage: T('4 * 2^m * m^4 * log(n)').variable_names()
('m', 'n')
sage: T('4 * 2^m * m^4').variable_names()
('m',)
sage: T('4 * log(n)').variable_names()
('n',)
sage: T('4 * m^3').variable_names()
('m',)
sage: T('4 * m^0').variable_names()
()
```

class sage.rings.asymptotic.term_monoid.**GenericTermMonoid**(*term_monoid_factory*, *growth_group*, *coefficient_ring*, *category*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent, *sage.rings.asymptotic.misc.WithLocals*

Parent for generic asymptotic terms.

INPUT:

- `growth_group` – a growth group (i.e. an instance of *GenericGrowthGroup*).
- `coefficient_ring` – a ring which contains the (maybe implicit) coefficients of the elements.
- `category` – The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of `Join of Category of Monoids and Category of posets`. This is also the default category if `None` is specified.

In this class the base structure for asymptotic term monoids will be handled. These monoids are the parents of asymptotic terms (for example, see *GenericTerm* or *OTerm*). Basically, asymptotic terms consist of a `growth` (which is an asymptotic growth group element, for example *MonomialGrowthElement*); additional structure and properties are added by the classes inherited from *GenericTermMonoid*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G_x = GrowthGroup('x^ZZ'); x = G_x.gen()
sage: G_y = GrowthGroup('y^QQ'); y = G_y.gen()
sage: T_x_ZZ = GenericTermMonoid(TermMonoid, G_x, QQ)
sage: T_y_QQ = GenericTermMonoid(TermMonoid, G_y, QQ)
sage: T_x_ZZ
GenericTerm Monoid x^ZZ with (implicit) coefficients in Rational Field
sage: T_y_QQ
GenericTerm Monoid y^QQ with (implicit) coefficients in Rational Field
```

Element
alias of *GenericTerm*

change_parameter(*growth_group=None*, *coefficient_ring=None*)
Return a term monoid with a change in one or more of the given parameters.

INPUT:

- `growth_group` – (default: `None`) the new growth group.
- `coefficient_ring` – (default: `None`) the new coefficient ring.

OUTPUT:

A term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: E = TermMonoid('exact', GrowthGroup('n^ZZ'), ZZ)
```

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```
sage: E.change_parameter(coefficient_ring=QQ)
Exact Term Monoid n^ZZ with coefficients in Rational Field
sage: E.change_parameter(growth_group=GrowthGroup('n^QQ'))
Exact Term Monoid n^QQ with coefficients in Integer Ring
```

coefficient_ring

The coefficient ring of this term monoid, i.e. the ring where the coefficients are from.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: GenericTermMonoid(TermMonoid, GrowthGroup('x^ZZ'), ZZ).coefficient_ring
Integer Ring
```

from_construction(*construction*, ***kwds_overrides*)

Create a term from the construction of another term.

INPUT:

- `construction` – a pair `(cls, kwds_construction)`
- `kwds_overrides` – a dictionary

OUTPUT:

A term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: x = G.gen()
sage: T = TermMonoid('O', G, QQ)
sage: o = T.an_element()
```

We use a construction directly as input:

```
sage: T.from_construction(o.construction())
O(x)
```

We can override the given data:

```
sage: T.from_construction(o.construction(), growth=x^2)
O(x^2)
```

A minimalistic example:

```
sage: T.from_construction((None, {'growth': x}))
O(x)
```

See also:

GenericTerm.construction(), *TermWithCoefficient.construction()*

growth_group

The growth group underlying this term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ).growth_group
Growth Group x^ZZ
```

le(*left*, *right*)

Return whether the term `left` is at most (less than or equal to) the term `right`.

INPUT:

- `left` – an element.
- `right` – an element.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import GenericTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: T = GenericTermMonoid(TermMonoid, G, QQ)
sage: t1 = T(x); t2 = T(x^2)
sage: T.le(t1, t2)
True
```

some_elements()

Return some elements of this term monoid.

See `TestSuite` for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: tuple(TermMonoid('O', G, QQ).some_elements())
(O(1), O(x), O(x^(-1)), O(x^2), O(x^(-2)), O(x^3), ...)
```

term_monoid(*type*)

Return the term monoid of specified `type`.

INPUT:

- `type` – 'O' or 'exact', or an instance of an existing term monoid. See *TermMonoidFactory* for more details.

OUTPUT:

A term monoid object derived from *GenericTermMonoid*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: E = TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ); E
Exact Term Monoid x^ZZ with coefficients in Integer Ring
sage: E.term_monoid('O')
O-Term Monoid x^ZZ with implicit coefficients in Integer Ring
```

term_monoid_factory

The term monoid factory capable of creating this term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup

sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory
sage: DefaultTermMonoidFactory('exact', GrowthGroup('x^ZZ'), ZZ).term_monoid_
↪factory
Term Monoid Factory 'sage.rings.asymptotic.term_monoid.DefaultTermMonoidFactory'

sage: from sage.rings.asymptotic.term_monoid import TermMonoidFactory
sage: TermMonoid = TermMonoidFactory('__main__.TermMonoid')

sage: TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ).term_monoid_factory
Term Monoid Factory '__main__.TermMonoid'
```

class sage.rings.asymptotic.term_monoid.**OTerm**(*parent*, *growth*)

Bases: *sage.rings.asymptotic.term_monoid.GenericTerm*

Class for an asymptotic term representing an O-term with specified growth. For the mathematical properties of O-terms see Wikipedia article Big_O_Notation.

O-terms can *absorb* terms of weaker or equal growth.

INPUT:

- `parent` – the parent of the asymptotic term.
- `growth` – a growth element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import OTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
```

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```
sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: OT = OTermMonoid(TermMonoid, G, QQ)
sage: t1 = OT(x^-7); t2 = OT(x^5); t3 = OT(x^42)
sage: t1, t2, t3
(O(x^(-7)), O(x^5), O(x^42))
sage: t1.can_absorb(t2)
False
sage: t2.can_absorb(t1)
True
sage: t2.absorb(t1)
O(x^5)
sage: t1 <= t2 and t2 <= t3
True
sage: t3 <= t1
False
```

The conversion of growth elements also works for the creation of O-terms:

```
sage: x = SR('x'); x.parent()
Symbolic Ring
sage: OT(x^17)
O(x^17)
```

can_absorb(*other*)

Check whether this O-term can absorb `other`.

INPUT:

- `other` – an asymptotic term.

OUTPUT:

A boolean.

Note: An *OTerm* can absorb any other asymptotic term with weaker or equal growth.

See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: OT = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: t1 = OT(x^21); t2 = OT(x^42)
sage: t1.can_absorb(t2)
False
sage: t2.can_absorb(t1)
True
```

is_little_o_of_one()

Return whether this O-term is of order $o(1)$.

INPUT:

Nothing.

OUTPUT:

A boolean.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), QQ)
sage: T(x).is_little_o_of_one()
False
sage: T(1).is_little_o_of_one()
False
sage: T(x^(-1)).is_little_o_of_one()
True
```

```
sage: T = TermMonoid('O', GrowthGroup('x^ZZ * y^ZZ'), QQ)
sage: T('x * y^(-1)').is_little_o_of_one()
False
sage: T('x^(-1) * y').is_little_o_of_one()
False
sage: T('x^(-2) * y^(-3)').is_little_o_of_one()
True
```

```
sage: T = TermMonoid('O', GrowthGroup('x^QQ * log(x)^QQ'), QQ)
sage: T('x * log(x)^2').is_little_o_of_one()
False
sage: T('x^2 * log(x)^(-1234)').is_little_o_of_one()
False
sage: T('x^(-1) * log(x)^4242').is_little_o_of_one()
True
sage: T('x^(-1/100) * log(x)^(1000/7)').is_little_o_of_one()
True
```

log_term(*base=None*, *locals=None*)

Determine the logarithm of this O-term.

INPUT:

- `base` – the base of the logarithm. If `None` (default value) is used, the natural logarithm is taken.
- `locals` – a dictionary which may contain the following keys and values:
  - `'log'` – value: a function. If not used, then the usual `log` is taken.

OUTPUT:

A tuple of terms.

Note: This method returns a tuple with the summands that come from applying the rule $\log(x \cdot y) = \log(x) + \log(y)$.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T(x^2).log_term()
(O(log(x)),)
sage: T(x^1234).log_term()
(O(log(x)),)
```

```
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ * log(x)^ZZ * y^ZZ * log(y)^ZZ'),␣
↪QQ)
sage: T('x * y').log_term()
(O(log(x)), O(log(y)))
```

See also:

ExactTerm.log_term().

rpow(*base*)

Return the power of `base` to this O-term.

INPUT:

- `base` – an element or `'e'`.

OUTPUT:

A term.

Note: For *OTerm*, the powers can only be constructed for exponents $O(1)$ or if `base` is 1.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: T = TermMonoid('O', GrowthGroup('x^ZZ * log(x)^ZZ'), QQ)
sage: T(1).rpow('e')
O(1)
sage: T(1).rpow(2)
O(1)
```

```
sage: T.an_element().rpow(1)
1
sage: T('x^2').rpow(1)
1
```

```
sage: T.an_element().rpow('e')
Traceback (most recent call last):
...
ValueError: Cannot take e to the exponent O(x*log(x)) in
O-Term Monoid x^ZZ * log(x)^ZZ with implicit coefficients in Rational Field
sage: T('log(x)').rpow('e')
```

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```
Traceback (most recent call last):
...
ValueError: Cannot take e to the exponent O(log(x)) in
O-Term Monoid x^ZZ * log(x)^ZZ with implicit coefficients in Rational Field
```

class sage.rings.asymptotic.term_monoid.**OTermMonoid**(*term_monoid_factory*, *growth_group*, *coefficient_ring*, *category*)

Bases: *sage.rings.asymptotic.term_monoid.GenericTermMonoid*

Parent for asymptotic big O-terms.

INPUT:

- `growth_group` – a growth group.
- `category` – The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of `Join of Category of monoids and Category of posets`. This is also the default category if `None` is specified.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import OTermMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G_x_ZZ = GrowthGroup('x^ZZ')
sage: G_y_QQ = GrowthGroup('y^QQ')
sage: OT_x_ZZ = OTermMonoid(TermMonoid, G_x_ZZ, QQ); OT_x_ZZ
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: OT_y_QQ = OTermMonoid(TermMonoid, G_y_QQ, QQ); OT_y_QQ
O-Term Monoid y^QQ with implicit coefficients in Rational Field
```

O-term monoids can also be created by using the *term factory*:

```
sage: TermMonoid('O', G_x_ZZ, QQ) is OT_x_ZZ
True
sage: TermMonoid('O', GrowthGroup('x^QQ'), QQ)
O-Term Monoid x^QQ with implicit coefficients in Rational Field
```

Element
alias of *OTerm*

class sage.rings.asymptotic.term_monoid.**TermMonoidFactory**(*name*, *exact_term_monoid_class=None*, *O_term_monoid_class=None*, *B_term_monoid_class=None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.factory.UniqueFactory`

Factory for asymptotic term monoids. It can generate the following term monoids:

- *OTermMonoid*,
- *ExactTermMonoid*,
- *BTermMonoid*.

Note: An instance of this factory is available as `DefaultTermMonoidFactory`.

INPUT:

- `term_monoid` – the kind of terms held in the new term monoid. Either a string `'exact'`, `'O'` (capital letter O) or `'B'` or an existing instance of a term monoid.
- `growth_group` – a growth group or a string describing a growth group.
- `coefficient_ring` – a ring.
- `asymptotic_ring` – if specified, then `growth_group` and `coefficient_ring` are taken from this asymptotic ring.

OUTPUT:

An asymptotic term monoid.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: TermMonoid('O', G, QQ)
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: TermMonoid('exact', G, ZZ)
Exact Term Monoid x^ZZ with coefficients in Integer Ring
```

```
sage: R = AsymptoticRing(growth_group=G, coefficient_ring=QQ)
sage: TermMonoid('exact', asymptotic_ring=R)
Exact Term Monoid x^ZZ with coefficients in Rational Field
sage: TermMonoid('O', asymptotic_ring=R)
O-Term Monoid x^ZZ with implicit coefficients in Rational Field

sage: TermMonoid('exact', 'QQ^m * m^QQ * log(n)^ZZ', ZZ)
Exact Term Monoid QQ^m * m^QQ * Signs^m * log(n)^ZZ
with coefficients in Integer Ring
```

create_key_and_extra_args(*term_monoid*, *growth_group=None*, *coefficient_ring=None*, *asymptotic_ring=None*, ***kwds*)

Given the arguments and keyword, create a key that uniquely determines this object.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: TermMonoid.create_key_and_extra_args('O', G, QQ)
((<class 'sage.rings.asymptotic.term_monoid.OTermMonoid'>,
  Growth Group x^ZZ, Rational Field), {})
sage: TermMonoid.create_key_and_extra_args('exact', G, ZZ)
((<class 'sage.rings.asymptotic.term_monoid.ExactTermMonoid'>,
  Growth Group x^ZZ, Integer Ring), {})
sage: TermMonoid.create_key_and_extra_args('exact', G)
Traceback (most recent call last):
...
ValueError: A coefficient ring has to be specified
to create a term monoid of type 'exact'
```

create_object(*version*, *key*, ***kwds*)
Create a object from the given arguments.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: G = GrowthGroup('x^ZZ')
sage: TermMonoid('O', G, QQ)  # indirect doctest
O-Term Monoid x^ZZ with implicit coefficients in Rational Field
sage: TermMonoid('exact', G, ZZ)  # indirect doctest
Exact Term Monoid x^ZZ with coefficients in Integer Ring
```

class sage.rings.asymptotic.term_monoid.**TermWithCoefficient**(*parent*, *growth*, *coefficient*)
Bases: *sage.rings.asymptotic.term_monoid.GenericTerm*

Base class for asymptotic terms possessing a coefficient. For example, *ExactTerm* directly inherits from this class.

INPUT:

- `parent` – the parent of the asymptotic term.
- `growth` – an asymptotic growth element of the parent's growth group.
- `coefficient` – an element of the parent's coefficient ring.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G = GrowthGroup('x^ZZ'); x = G.gen()
sage: CT_ZZ = TermWithCoefficientMonoid(TermMonoid, G, ZZ)
sage: CT_QQ = TermWithCoefficientMonoid(TermMonoid, G, QQ)
sage: CT_ZZ(x^2, coefficient=5)
Term with coefficient 5 and growth x^2
sage: CT_QQ(x^3, coefficient=3/8)
Term with coefficient 3/8 and growth x^3
```

construction()
Return a construction of this term.

INPUT:

Nothing.

OUTPUT:

A pair `(cls, kwds)` such that `cls(**kwds)` equals this term.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
```

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```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ'), QQ)
sage: a = T.an_element(); a
1/2*x
sage: cls, kwds = a.construction(); cls, kwds
(<class 'sage.rings.asymptotic.term_monoid.ExactTermMonoid_with_category.
↪element_class'>,
 {'coefficient': 1/2,
  'growth': x,
  'parent': Exact Term Monoid x^ZZ with coefficients in Rational Field})
sage: cls(**kwds) == a
True
```

See also:

GenericTerm.construction(), *GenericTermMonoid.from_construction()*

class sage.rings.asymptotic.term_monoid.**TermWithCoefficientMonoid**(*term_monoid_factory*, *growth_group*, *coefficient_ring*, *category*)

Bases: *sage.rings.asymptotic.term_monoid.GenericTermMonoid*

This class implements the base structure for parents of asymptotic terms possessing a coefficient from some coefficient ring. In particular, this is also the parent for *TermWithCoefficient*.

INPUT:

- `growth_group` – a growth group.
- `category` – The category of the parent can be specified in order to broaden the base structure. It has to be a subcategory of `Join of Category of monoids and Category of posets`. This is also the default category if `None` is specified.
- `coefficient_ring` – the ring which contains the coefficients of the elements.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import TermWithCoefficientMonoid
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid

sage: G_ZZ = GrowthGroup('x^ZZ'); x_ZZ = G_ZZ.gen()
sage: G_QQ = GrowthGroup('x^QQ'); x_QQ = G_QQ.gen()
sage: TC_ZZ = TermWithCoefficientMonoid(TermMonoid, G_ZZ, QQ); TC_ZZ
TermWithCoefficient Monoid x^ZZ with coefficients in Rational Field
sage: TC_QQ = TermWithCoefficientMonoid(TermMonoid, G_QQ, QQ); TC_QQ
TermWithCoefficient Monoid x^QQ with coefficients in Rational Field
sage: TC_ZZ == TC_QQ or TC_ZZ is TC_QQ
False
sage: TC_QQ.coerce_map_from(TC_ZZ)
Coercion map:
  From: TermWithCoefficient Monoid x^ZZ with coefficients in Rational Field
  To:   TermWithCoefficient Monoid x^QQ with coefficients in Rational Field
```

Element
alias of *TermWithCoefficient*

some_elements()
Return some elements of this term with coefficient monoid.

See TestSuite for a typical use case.

INPUT:

Nothing.

OUTPUT:

An iterator.

EXAMPLES:

```
sage: from itertools import islice
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: G = GrowthGroup('z^QQ')
sage: T = TermMonoid('exact', G, ZZ)
sage: tuple(islice(T.some_elements(), int(10)))
(z^(1/2), z^(-1/2), -z^(1/2), z^2, -z^(-1/2), 2*z^(1/2),
 z^(-2), -z^2, 2*z^(-1/2), -2*z^(1/2))
```

exception sage.rings.asymptotic.term_monoid.**ZeroCoefficientError**
Bases: ValueError

sage.rings.asymptotic.term_monoid.**absorption**(*left*, *right*)
Let one of the two passed terms absorb the other.

Helper function used by *AsymptoticExpansion*.

Note: If neither of the terms can absorb the other, an ArithmeticError is raised.

See the *module description* for a detailed explanation of absorption.

INPUT:

- left – an asymptotic term.
- right – an asymptotic term.

OUTPUT:

An asymptotic term or None.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: from sage.rings.asymptotic.term_monoid import absorption
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), ZZ)
sage: absorption(T(x^2), T(x^3))
O(x^3)
sage: absorption(T(x^3), T(x^2))
O(x^3)
```

```
sage: T = TermMonoid('exact', GrowthGroup('x^ZZ'), ZZ)
sage: absorption(T(x^2), T(x^3))
Traceback (most recent call last):
...
ArithmeticError: Absorption between x^2 and x^3 is not possible.
```

sage.rings.asymptotic.term_monoid.**can_absorb**(*left*, *right*)

Return whether one of the two input terms is able to absorb the other.

Helper function used by *AsymptoticExpansion*.

INPUT:

- `left` – an asymptotic term.
- `right` – an asymptotic term.

OUTPUT:

A boolean.

Note: See the *module description* for a detailed explanation of absorption.

EXAMPLES:

```
sage: from sage.rings.asymptotic.growth_group import GrowthGroup
sage: from sage.rings.asymptotic.term_monoid import DefaultTermMonoidFactory as␣
↪TermMonoid
sage: from sage.rings.asymptotic.term_monoid import can_absorb
sage: T = TermMonoid('O', GrowthGroup('x^ZZ'), ZZ)
sage: can_absorb(T(x^2), T(x^3))
True
sage: can_absorb(T(x^3), T(x^2))
True
```

4.6 Asymptotic Expansions — Miscellaneous

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4.6.1 Functions, Classes and Methods

class sage.rings.asymptotic.misc.**Locals**

Bases: dict

A frozen dictionary-like class for storing locals of an *AsymptoticRing*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import Locals
sage: locals = Locals({'a': 42})
sage: locals['a']
42
```

The object contains default values (see *default_locals()*) for some keys:

```
sage: locals['log']
<function log at 0x...>
```

default_locals()

Return the default locals used in the *AsymptoticRing*.

OUTPUT:

A dictionary.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import Locals
sage: locals = Locals({'a': 2, 'b': 1})
sage: locals
{'a': 2, 'b': 1}
sage: locals.default_locals()
{'log': <function log at 0x...>}
sage: locals['log']
<function log at 0x...>
```

exception sage.rings.asymptotic.misc.**NotImplementedBZero**(*asymptotic_ring=None*, *var=None*, *exact_part=0*)

Bases: NotImplementedError

A special NotImplementedError which is raised when the result is B(0) which means 0 for sufficiently large values of the variable.

exception sage.rings.asymptotic.misc.**NotImplementedOZero**(*asymptotic_ring=None*, *var=None*, *exact_part=0*)

Bases: NotImplementedError

A special NotImplementedError which is raised when the result is O(0) which means 0 for sufficiently large values of the variable.

class sage.rings.asymptotic.misc.**WithLocals**

Bases: sage.structure.sage_object.SageObject

A class extensions for handling local values; see also *Locals*.

This is used in the *AsymptoticRing*.

EXAMPLES:

```
sage: A.<n> = AsymptoticRing('n^ZZ', QQ, locals={'a': 42})
sage: A.locals()
{'a': 42}
```

locals(*locals=None*)

Return the actual *Locals* object to be used.

INPUT:

- `locals` – an object

  If `locals` is not `None`, then a *Locals* object is created and returned. If `locals` is `None`, then a stored *Locals* object, if any, is returned. Otherwise, an empty (i.e. no values except the default values) *Locals* object is created and returned.

OUTPUT:

A *Locals* object.

sage.rings.asymptotic.misc.**bidirectional_merge_overlapping**(*A*, *B*, *key=None*)

Merge the two overlapping tuples/lists.

INPUT:

- `A` – a list or tuple (type has to coincide with type of `B`).
- `B` – a list or tuple (type has to coincide with type of `A`).
- `key` – (default: `None`) a function. If `None`, then the identity is used. This `key`-function applied on an element of the list/tuple is used for comparison. Thus elements with the same key are considered as equal.

OUTPUT:

A pair of lists or tuples (depending on the type of `A` and `B`).

Note: Suppose we can decompose the list $A = ac$ and $B = cb$ with lists a, b, c, where c is nonempty. Then *bidirectional_merge_overlapping()* returns the pair (acb, acb).

Suppose a `key`-function is specified and $A = ac_A$ and $B = c_B b$, where the list of keys of the elements of c_A equals the list of keys of the elements of c_B. Then *bidirectional_merge_overlapping()* returns the pair $(ac_A b, ac_B b)$.

After unsuccessfully merging $A = ac$ and $B = cb$, a merge of $A = ca$ and $B = bc$ is tried.

---

sage.rings.asymptotic.misc.**bidirectional_merge_sorted**(*A*, *B*, *key=None*)

Merge the two tuples/lists, keeping the orders provided by them.

INPUT:

- `A` – a list or tuple (type has to coincide with type of `B`).
- `B` – a list or tuple (type has to coincide with type of `A`).
- `key` – (default: `None`) a function. If `None`, then the identity is used. This `key`-function applied on an element of the list/tuple is used for comparison. Thus elements with the same key are considered as equal.

Note: The two tuples/list need to overlap, i.e. need at least one key in common.

OUTPUT:

A pair of lists containing all elements totally ordered. (The first component uses `A` as a merge base, the second component `B`.)

If merging fails, then a RuntimeError is raised.

sage.rings.asymptotic.misc.**combine_exceptions**(*e*, **f*)
Helper function which combines the messages of the given exceptions.

INPUT:

- `e` – an exception.
- `*f` – exceptions.

OUTPUT:

An exception.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import combine_exceptions
sage: raise combine_exceptions(ValueError('Outer.'), TypeError('Inner.'))
Traceback (most recent call last):
...
ValueError: Outer.
> *previous* TypeError: Inner.
sage: raise combine_exceptions(ValueError('Outer.'),
....:                          TypeError('Inner1.'), TypeError('Inner2.'))
Traceback (most recent call last):
...
ValueError: Outer.
> *previous* TypeError: Inner1.
> *and* TypeError: Inner2.
sage: raise combine_exceptions(ValueError('Outer.'),
....:                          combine_exceptions(TypeError('Middle.'),
....:                                             TypeError('Inner.')))
Traceback (most recent call last):
...
ValueError: Outer.
> *previous* TypeError: Middle.
>> *previous* TypeError: Inner.
```

sage.rings.asymptotic.misc.**log_string**(*element*, *base=None*)
Return a representation of the log of the given element to the given base.

INPUT:

- `element` – an object.
- `base` – an object or `None`.

OUTPUT:

A string.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import log_string
sage: log_string(3)
'log(3)'
```

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```
sage: log_string(3, base=42)
'log(3, base=42)'
```

sage.rings.asymptotic.misc.**parent_to_repr_short**(*P*)

Helper method which generates a short(er) representation string out of a parent.

INPUT:

- P – a parent.

OUTPUT:

A string.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import parent_to_repr_short
sage: parent_to_repr_short(ZZ)
'ZZ'
sage: parent_to_repr_short(QQ)
'QQ'
sage: parent_to_repr_short(SR)
'SR'
sage: parent_to_repr_short(RR)
'RR'
sage: parent_to_repr_short(CC)
'CC'
sage: parent_to_repr_short(ZZ['x'])
'ZZ[x]'
sage: parent_to_repr_short(QQ['d, k'])
'QQ[d, k]'
sage: parent_to_repr_short(QQ['e'])
'QQ[e]'
sage: parent_to_repr_short(SR[['a, r']])
'SR[[a, r]]'
sage: parent_to_repr_short(Zmod(3))
'Ring of integers modulo 3'
sage: parent_to_repr_short(Zmod(3)['g'])
'Univariate Polynomial Ring in g over Ring of integers modulo 3'
```

sage.rings.asymptotic.misc.**repr_op**(*left*, *op*, *right=None*, *latex=False*)

Create a string `left op right` with taking care of parentheses in its operands.

INPUT:

- `left` – an element.
- `op` – a string.
- `right` – an element.
- `latex` – (default: `False`) a boolean. If set, then LaTeX-output is returned.

OUTPUT:

A string.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import repr_op
sage: repr_op('a^b', '^', 'c')
'(a^b)^c'
```

sage.rings.asymptotic.misc.**repr_short_to_parent**(*s*)
Helper method for the growth group factory, which converts a short representation string to a parent.

INPUT:

- `s` – a string, short representation of a parent.

OUTPUT:

A parent.

The possible short representations are shown in the examples below.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import repr_short_to_parent
sage: repr_short_to_parent('ZZ')
Integer Ring
sage: repr_short_to_parent('QQ')
Rational Field
sage: repr_short_to_parent('SR')
Symbolic Ring
sage: repr_short_to_parent('NN')
Non negative integer semiring
sage: repr_short_to_parent('UU')
Group of Roots of Unity
```

sage.rings.asymptotic.misc.**split_str_by_op**(*string*, *op*, *strip_parentheses=True*)
Split the given string into a tuple of substrings arising by splitting by `op` and taking care of parentheses.

INPUT:

- `string` – a string.
- `op` – a string. This is used by str.split. Thus, if this is `None`, then any whitespace string is a separator and empty strings are removed from the result.
- `strip_parentheses` – (default: `True`) a boolean.

OUTPUT:

A tuple of strings.

sage.rings.asymptotic.misc.**strip_symbolic**(*expression*)
Return, if possible, the underlying (numeric) object of the symbolic expression.

If `expression` is not symbolic, then `expression` is returned.

INPUT:

- `expression` – an object

OUTPUT:

An object.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import strip_symbolic
sage: strip_symbolic(SR(2)); _.parent()
2
Integer Ring
sage: strip_symbolic(SR(2/3)); _.parent()
2/3
Rational Field
sage: strip_symbolic(SR('x')); _.parent()
x
Symbolic Ring
sage: strip_symbolic(pi); _.parent()
pi
Symbolic Ring
```

sage.rings.asymptotic.misc.**substitute_raise_exception**(*element*, *e*)

Raise an error describing what went wrong with the substitution.

INPUT:

- `element` – an element.
- `e` – an exception which is included in the raised error message.

OUTPUT:

Raise an exception of the same type as `e`.

sage.rings.asymptotic.misc.**transform_category**(*category*, *subcategory_mapping*, *axiom_mapping*, *initial_category=None*)

Transform `category` to a new category according to the given mappings.

INPUT:

- `category` – a category.
- `subcategory_mapping` – a list (or other iterable) of triples `(from, to, mandatory)`, where
  - `from` and `to` are categories and
  - `mandatory` is a boolean.
- `axiom_mapping` – a list (or other iterable) of triples `(from, to, mandatory)`, where
  - `from` and `to` are strings describing axioms and
  - `mandatory` is a boolean.
- `initial_category` – (default: `None`) a category. When transforming the given category, this `initial_category` is used as a starting point of the result. This means the resulting category will be a subcategory of `initial_category`. If `initial_category` is `None`, then the `category of objects` is used.

OUTPUT:

A category.

Note: Consider a subcategory mapping `(from, to, mandatory)`. If `category` is a subcategory of `from`, then the returned category will be a subcategory of `to`. Otherwise and if `mandatory` is set, then an error is raised.

Consider an axiom mapping `(from, to, mandatory)`. If `category` is has axiom `from`, then the returned category will have axiom `to`. Otherwise and if `mandatory` is set, then an error is raised.

EXAMPLES:

```
sage: from sage.rings.asymptotic.misc import transform_category
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: from sage.categories.additive_groups import AdditiveGroups
sage: S = [
....:     (Sets(), Sets(), True),
....:     (Posets(), Posets(), False),
....:     (AdditiveMagmas(), Magmas(), False)]
sage: A = [
....:     ('AdditiveAssociative', 'Associative', False),
....:     ('AdditiveUnital', 'Unital', False),
....:     ('AdditiveInverse', 'Inverse', False),
....:     ('AdditiveCommutative', 'Commutative', False)]
sage: transform_category(Objects(), S, A)
Traceback (most recent call last):
...
ValueError: Category of objects is not
a subcategory of Category of sets.
sage: transform_category(Sets(), S, A)
Category of sets
sage: transform_category(Posets(), S, A)
Category of posets
sage: transform_category(AdditiveSemigroups(), S, A)
Category of semigroups
sage: transform_category(AdditiveMonoids(), S, A)
Category of monoids
sage: transform_category(AdditiveGroups(), S, A)
Category of groups
sage: transform_category(AdditiveGroups().AdditiveCommutative(), S, A)
Category of commutative groups
```

```
sage: transform_category(AdditiveGroups().AdditiveCommutative(), S, A,
....:     initial_category=Posets())
Join of Category of commutative groups
    and Category of posets
```

```
sage: transform_category(ZZ.category(), S, A)
Category of commutative groups
sage: transform_category(QQ.category(), S, A)
Category of commutative groups
sage: transform_category(SR.category(), S, A)
Category of commutative groups
sage: transform_category(Fields(), S, A)
Category of commutative groups
sage: transform_category(ZZ['t'].category(), S, A)
Category of commutative groups
```

```
sage: A[-1] = ('Commutative', 'AdditiveCommutative', True)
sage: transform_category(Groups(), S, A)
Traceback (most recent call last):
...
ValueError: Category of groups does not have
axiom Commutative.
```

4.7 Asymptotics of Multivariate Generating Series

Let $F(x) = \sum_{\nu \in \mathbf{N}^d} F_\nu x^\nu$ be a multivariate power series with complex coefficients that converges in a neighborhood of the origin. Assume that $F = G/H$ for some functions G and H holomorphic in a neighborhood of the origin. Assume also that H is a polynomial.

This computes asymptotics for the coefficients $F_{r\alpha}$ as $r \to \infty$ with $r\alpha \in \mathbf{N}^d$ for α in a permissible subset of d-tuples of positive reals. More specifically, it computes arbitrary terms of the asymptotic expansion for $F_{r\alpha}$ when the asymptotics are controlled by a strictly minimal multiple point of the algebraic variety $H = 0$.

The algorithms and formulas implemented here come from [RW2008] and [RW2012]. For a general reference take a look in the book [PW2013].

4.7.1 Introductory Examples

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions import␣
↪FractionWithFactoredDenominatorRing
```

A univariate smooth point example:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (x - 1/2)^3
sage: Hfac = H.factor()
sage: G = -1/(x + 3)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-1/(x + 3), [(x - 1/2, 3)])
sage: alpha = [1]
sage: decomp = F.asymptotic_decomposition(alpha)
sage: decomp
(0, []) +
(-1/2*r^2*(x^2/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
 + 6*x/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
 + 9/(x^5 + 9*x^4 + 27*x^3 + 27*x^2))
 - 1/2*r*(5*x^2/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
 + 24*x/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
 + 27/(x^5 + 9*x^4 + 27*x^3 + 27*x^2))
 - 3*x^2/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
 - 9*x/(x^5 + 9*x^4 + 27*x^3 + 27*x^2)
 - 9/(x^5 + 9*x^4 + 27*x^3 + 27*x^2),
 [(x - 1/2, 1)])
sage: F1 = decomp[1]
```

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```
sage: p = {x: 1/2}
sage: asy = F1.asymptotics(p, alpha, 3)
sage: asy
(8/343*(49*r^2 + 161*r + 114)*2^r, 2, 8/7*r^2 + 184/49*r + 912/343)
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8, 16], asy[1])
[((1,), 7.555555556, [7.556851312], [-0.0001714971672]),
 ((2,), 14.74074074, [14.74052478], [0.00001465051901]),
 ((4,), 35.96502058, [35.96501458], [1.667911934e-7]),
 ((8,), 105.8425656, [105.8425656], [4.399565380e-11]),
 ((16,), 355.3119534, [355.3119534], [0.0000000000])]
```

Another smooth point example (Example 5.4 of [RW2008]):

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: q = 1/2
sage: qq = q.denominator()
sage: H = 1 - q*x + q*x*y - x^2*y
sage: Hfac = H.factor()
sage: G = (1 - q*x)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = list(qq*vector([2, 1 - q]))
sage: alpha
[4, 1]
sage: I = F.smooth_critical_ideal(alpha)
sage: I
Ideal (y^2 - 2*y + 1, x + 1/4*y - 5/4) of
 Multivariate Polynomial Ring in x, y over Rational Field
sage: s = solve([SR(z) for z in I.gens()],
....:           [SR(z) for z in R.gens()], solution_dict=true)
sage: s == [{SR(x): 1, SR(y): 1}]
True
sage: p = s[0]
sage: asy = F.asymptotics(p, alpha, 1, verbose=True)
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
sage: asy
(1/24*2^(2/3)*(sqrt(3) + 4/(sqrt(3) + I) + I)*gamma(1/3)/(pi*r^(1/3)),
 1,
 1/24*2^(2/3)*(sqrt(3) + 4/(sqrt(3) + I) + I)*gamma(1/3)/(pi*r^(1/3)))
sage: r = SR('r')
sage: tuple((a*r^(1/3)).full_simplify() / r^(1/3) for a in asy)  # make nicer␣
↪coefficients
(1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3)),
 1,
 1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3)))
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8, 16], asy[1])
[((4, 1), 0.1875000000, [0.1953794675...], [-0.042023826...]),
 ((8, 2), 0.1523437500, [0.1550727862...], [-0.017913673...]),
 ((16, 4), 0.1221771240, [0.1230813519...], [-0.0074009592...]),
```

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```
 ((32, 8), 0.09739671811, [0.09768973377...], [-0.0030084757...]),
 ((64, 16), 0.07744253816, [0.07753639308...], [-0.0012119297...])]
```

A multiple point example (Example 6.5 of [RW2012]):

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - 2*x - y)**2 * (1 - x - 2*y)**2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(1, [(x + 2*y - 1, 2), (2*x + y - 1, 2)])
sage: I = F.singular_ideal()
sage: I
Ideal (x - 1/3, y - 1/3) of
 Multivariate Polynomial Ring in x, y over Rational Field
sage: p = {x: 1/3, y: 1/3}
sage: F.is_convenient_multiple_point(p)
(True, 'convenient in variables [x, y]')
sage: alpha = (var('a'), var('b'))
sage: decomp =  F.asymptotic_decomposition(alpha); decomp
(0, []) +
(-1/9*r^2*(2*a^2/x^2 + 2*b^2/y^2 - 5*a*b/(x*y))
 - 1/9*r*(6*a/x^2 + 6*b/y^2 - 5*a/(x*y) - 5*b/(x*y))
 - 4/9/x^2 - 4/9/y^2 + 5/9/(x*y),
 [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: F1 = decomp[1]
sage: F1.asymptotics(p, alpha, 2)
(-3*((2*a^2 - 5*a*b + 2*b^2)*r^2 + (a + b)*r + 3)*(1/((1/3)^a*(1/3)^b))^r,
 1/((1/3)^a*(1/3)^b), -3*(2*a^2 - 5*a*b + 2*b^2)*r^2 - 3*(a + b)*r - 9)
sage: alpha = [4, 3]
sage: decomp =  F.asymptotic_decomposition(alpha)
sage: F1 = decomp[1]
sage: asy = F1.asymptotics(p, alpha, 2)
sage: asy
(3*(10*r^2 - 7*r - 3)*2187^r, 2187, 30*r^2 - 21*r - 9)
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8], asy[1])
[((4, 3), 30.72702332, [0.0000000000], [1.000000000]),
 ((8, 6), 111.9315678, [69.00000000], [0.3835519207]),
 ((16, 12), 442.7813138, [387.0000000], [0.1259793763]),
 ((32, 24), 1799.879232, [1743.000000], [0.03160169385])]
```

4.7.2 Various

AUTHORS:

- Alexander Raichev (2008)
- Daniel Krenn (2014, 2016)

4.7.3 Classes and Methods

class sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**FractionWithFactoredDenominat**

Bases: sage.structure.element.RingElement

This element represents a fraction with a factored polynomial denominator. See also its parent *FractionWithFactoredDenominatorRing* for details.

Represents a fraction with factored polynomial denominator (FFPD) $p/(q_1^{e_1} \cdots q_n^{e_n})$ by storing the parts p and $[(q_1, e_1), \ldots, (q_n, e_n)]$. Here $q_1, \ldots, q_n$ are elements of a 0- or multi-variate factorial polynomial ring R, $q_1, \ldots, q_n$ are distinct irreducible elements of R, $e_1, \ldots, e_n$ are positive integers, and p is a function of the indeterminates of R (e.g., a Sage symbolic expression). An element r with no polynomial denominator is represented as `(r, [])`.

INPUT:

- `numerator` – an element p; this can be of any ring from which parent's base has coercion in
- `denominator_factored` – a list of the form $[(q_1, e_1), \ldots, (q_n, e_n)]$, where the $q_1, \ldots, q_n$ are distinct irreducible elements of R and the e_i are positive integers
- `reduce` – (optional) if `True`, then represent $p/(q_1^{e_1} \cdots q_n^{e_n})$ in lowest terms, otherwise this won't attempt to divide p by any of the q_i

OUTPUT:

An element representing the rational expression $p/(q_1^{e_1} \cdots q_n^{e_n})$.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: df = [x, 1], [y, 1], [x*y+1, 1]
sage: f = FFPD(x, df)
sage: f
(1, [(y, 1), (x*y + 1, 1)])
sage: ff = FFPD(x, df, reduce=False)
```

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```
sage: ff
(x, [(y, 1), (x, 1), (x*y + 1, 1)])

sage: f = FFPD(x + y, [(x + y, 1)])
sage: f
(1, [])
```

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: FFPD(f)
(5*x^7 - 5*x^6 + 5/3*x^5 - 5/3*x^4 + 2*x^3 - 2/3*x^2 + 1/3*x - 1/3,
[(x - 1, 1), (x, 1), (x^2 + 1/3, 1)])
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = 2*y/(5*(x^3 - 1)*(y + 1))
sage: FFPD(f)
(2/5*y, [(y + 1, 1), (x - 1, 1), (x^2 + x + 1, 1)])

sage: p = 1/x^2
sage: q = 3*x**2*y
sage: qs = q.factor()
sage: f = FFPD(p/qs.unit(), qs)
sage: f
(1/3/x^2, [(y, 1), (x, 2)])

sage: f = FFPD(cos(x)*x*y^2, [(x, 2), (y, 1)])
sage: f
(x*y^2*cos(x), [(y, 1), (x, 2)])

sage: G = exp(x + y)
sage: H = (1 - 2*x - y) * (1 - x - 2*y)
sage: a = FFPD(G/H)
sage: a
(e^(x + y), [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: a.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
sage: b = FFPD(G, H.factor())
sage: b
(e^(x + y), [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: b.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
```

Singular throws a 'not implemented' error when trying to factor in a multivariate polynomial ring over an inexact field:

```
sage: R.<x,y> = PolynomialRing(CC)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = (x + 1)/(x*y*(x*y + 1)^2)
sage: FFPD(f)
Traceback (most recent call last):
```

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```
...
TypeError: Singular error:
   ? not implemented
   ? error occurred in or before STDIN line ...:
   `def sage...=factorize(sage...);`
```

AUTHORS:

- Alexander Raichev (2012-07-26)
- Daniel Krenn (2014-12-01)

algebraic_dependence_certificate()

Return the algebraic dependence certificate of `self`.

The algebraic dependence certificate is the ideal J of annihilating polynomials for the set of polynomials `[q^e for (q, e) in self.denominator_factored()]`, which could be the zero ideal. The ideal J lies in a polynomial ring over the field `self.denominator_ring.base_ring()` that has `m = len(self.denominator_factored())` indeterminates.

OUTPUT:

An ideal.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x^2 * (x*y + 1) * y^3)
sage: ff = FFPD(f)
sage: J = ff.algebraic_dependence_certificate(); J
Ideal (1 - 6*T2 + 15*T2^2 - 20*T2^3 + 15*T2^4 - T0^2*T1^3 -
 6*T2^5  + T2^6) of Multivariate Polynomial Ring in
 T0, T1, T2 over Rational Field
sage: g = J.gens()[0]
sage: df = ff.denominator_factored()
sage: g(*(q**e for q, e in df)) == 0
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = exp(x + y)
sage: H = x^2 * (x*y + 1) * y^3
sage: ff = FFPD(G, H.factor())
sage: J = ff.algebraic_dependence_certificate(); J
Ideal (1 - 6*T2 + 15*T2^2 - 20*T2^3 + 15*T2^4 - T0^2*T1^3 -
6*T2^5 + T2^6) of Multivariate Polynomial Ring in
T0, T1, T2 over Rational Field
sage: g = J.gens()[0]
sage: df = ff.denominator_factored()
sage: g(*(q**e for q, e in df)) == 0
True
```

```
sage: f = 1/(x^3 * y^2)
sage: J = FFPD(f).algebraic_dependence_certificate()
sage: J
Ideal (0) of Multivariate Polynomial Ring in T0, T1 over Rational Field
```

```
sage: f = sin(1)/(x^3 * y^2)
sage: J = FFPD(f).algebraic_dependence_certificate()
sage: J
Ideal (0) of Multivariate Polynomial Ring in T0, T1 over Rational Field
```

algebraic_dependence_decomposition(*whole_and_parts=True*)

Return an algebraic dependence decomposition of `self`.

Let $f = p/q$ where q lies in a d-variate polynomial ring $K[X]$ for some field K. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in $K[X]$ into irreducible factors and let V_i be the algebraic variety $\{x \in L^d \mid q_i(x) = 0\}$ of q_i over the algebraic closure L of K. By [Rai2012], f can be written as

$$(*) \quad \sum_A \frac{p_A}{\prod_{i \in A} q_i^{b_i}},$$

where the b_i are positive integers, each p_A is a products of p and an element in $K[X]$, and the sum is taken over all subsets $A \subseteq \{1, \ldots, m\}$ such that $|A| \leq d$ and $\{q_i \mid i \in A\}$ is algebraically independent.

We call $(*)$ an *algebraic dependence decomposition* of f. Algebraic dependence decompositions are not unique.

The algorithm used comes from [Rai2012].

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x^2 * (x*y + 1) * y^3)
sage: ff = FFPD(f)
sage: decomp = ff.algebraic_dependence_decomposition()
sage: decomp
(0, []) + (-x, [(x*y + 1, 1)]) +
(x^2*y^2 - x*y + 1, [(y, 3), (x, 2)])
sage: decomp.sum().quotient() == f
True
sage: for r in decomp:
....:     J = r.algebraic_dependence_certificate()
....:     J is None or J == J.ring().ideal()  # The zero ideal
True
True
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = sin(x)
```

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```
sage: H = x^2 * (x*y + 1) * y^3
sage: f = FFPD(G, H.factor())
sage: decomp = f.algebraic_dependence_decomposition()
sage: decomp
(0, []) + (x^4*y^3*sin(x), [(x*y + 1, 1)]) +
(-(x^5*y^5 - x^4*y^4 + x^3*y^3 - x^2*y^2 + x*y - 1)*sin(x),
 [(y, 3), (x, 2)])
sage: bool(decomp.sum().quotient() == G/H)
True
sage: for r in decomp:
....:     J = r.algebraic_dependence_certificate()
....:     J is None or J == J.ring().ideal()
True
True
True
```

asymptotic_decomposition(*alpha*, *asy_var=None*)

Return the asymptotic decomposition of `self`.

The asymptotic decomposition of F is a sum that has the same asymptotic expansion as f in the direction `alpha` but each summand has a denominator factorization of the form $[(q_1, 1), \ldots, (q_n, 1)]$, where n is at most the *dimension()* of F.

INPUT:

- `alpha` – a d-tuple of positive integers or symbolic variables
- `asy_var` – (default: `None`) a symbolic variable with respect to which to compute asymptotics; if `None` is given, we set `asy_var = var('r')`

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

The output results from a Leinartas decomposition followed by a cohomology decomposition.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = (x^2 + 1)/((x - 1)^3*(x + 2))
sage: F = FFPD(f)
sage: alpha = [var('a')]
sage: F.asymptotic_decomposition(alpha)
(0, []) +
(1/54*(5*a^2 + 2*a^2/x + 11*a^2/x^2)*r^2
 - 1/54*(5*a - 2*a/x - 33*a/x^2)*r + 11/27/x^2,
[(x - 1, 1)]) + (-5/27, [(x + 2, 1)])
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - 2*x -y)*(1 - x -2*y)**2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
```

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```
sage: F = FFPD(G, Hfac)
sage: alpha = var('a, b')
sage: F.asymptotic_decomposition(alpha)
(0, []) +
(-1/3*r*(a/x - 2*b/y) - 1/3/x + 2/3/y,
 [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
```

asymptotics(*p*, *alpha*, *N*, *asy_var=None*, *numerical=0*, *verbose=False*)

Return the asymptotics in the given direction.

This function returns the first N terms (some of which could be zero) of the asymptotic expansion of the Maclaurin ray coefficients $F_{r\alpha}$ of the function F represented by `self` as $r \to \infty$, where r is `asy_var` and `alpha` is a tuple of positive integers of length d which is `self.dimension()`. Assume that

- F is holomorphic in a neighborhood of the origin;
- the unique factorization of the denominator H of F in the local algebraic ring at p equals its unique factorization in the local analytic ring at p;
- the unique factorization of H in the local algebraic ring at p has at most `d` irreducible factors, none of which are repeated (one can reduce to this case via *asymptotic_decomposition()*);
- p is a convenient strictly minimal smooth or multiple point with all nonzero coordinates that is critical and nondegenerate for `alpha`.

The algorithms used here come from [RW2008] and [RW2012].

INPUT:

- `p` – a dictionary with keys that can be coerced to equal `self.denominator_ring.gens()`
- `alpha` – a tuple of length `self.dimension()` of positive integers or, if p is a smooth point, possibly of symbolic variables
- `N` – a positive integer
- `asy_var` – (default: `None`) a symbolic variable for the asymptotic expansion; if `none` is given, then `var('r')` will be assigned
- `numerical` – (default: 0) a natural number; if `numerical` is greater than 0, then return a numerical approximation of $F_{r\alpha}$ with `numerical` digits of precision; otherwise return exact values
- `verbose` – (default: `False`) print the current state of the algorithm

OUTPUT:

The tuple `(asy, exp_scale, subexp_part)`. Here `asy` is the sum of the first N terms (some of which might be 0) of the asymptotic expansion of $F_{r\alpha}$ as $r \to \infty$; `exp_scale**r` is the exponential factor of `asy`; `subexp_part` is the subexponential factor of `asy`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
```

A smooth point example:

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2
```

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```
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac); print(F)
(1, [(x*y + x + y - 1, 2)])
sage: alpha = [4, 3]
sage: decomp = F.asymptotic_decomposition(alpha); decomp
(0, []) + (... - 1/2, [(x*y + x + y - 1, 1)])
sage: F1 = decomp[1]
sage: p = {y: 1/3, x: 1/2}
sage: asy = F1.asymptotics(p, alpha, 2, verbose=True)
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
sage: asy
(1/6000*(3600*sqrt(5)*sqrt(3)*sqrt(2)*sqrt(r)/sqrt(pi)
  + 463*sqrt(5)*sqrt(3)*sqrt(2)/(sqrt(pi)*sqrt(r)))*432^r,
 432,
 3/5*sqrt(5)*sqrt(3)*sqrt(2)*sqrt(r)/sqrt(pi)
  + 463/6000*sqrt(5)*sqrt(3)*sqrt(2)/(sqrt(pi)*sqrt(r)))
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8, 16], asy[1])  # abs tol 1e-10
[((4, 3), 2.083333333, [2.092576110], [-0.004436533009]),
 ((8, 6), 2.787374614, [2.790732875], [-0.001204811281]),
 ((16, 12), 3.826259447, [3.827462310], [-0.0003143703383]),
 ((32, 24), 5.328112821, [5.328540787], [-0.00008032230388]),
 ((64, 48), 7.475927885, [7.476079664], [-0.00002030232879])]
```

A multiple point example:

```
sage: R.<x,y,z>= PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (4 - 2*x - y - z)**2*(4 - x - 2*y - z)
sage: Hfac = H.factor()
sage: G = 16/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-16, [(x + 2*y + z - 4, 1), (2*x + y + z - 4, 2)])
sage: alpha = [3, 3, 2]
sage: decomp = F.asymptotic_decomposition(alpha); decomp
(0, []) +
(16*r*(3/x - 2/z) + 16/x - 16/z,
 [(x + 2*y + z - 4, 1), (2*x + y + z - 4, 1)])
sage: F1 = decomp[1]
sage: p = {x: 1, y: 1, z: 1}
sage: asy = F1.asymptotics(p, alpha, 2, verbose=True) # long time
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second-order differential operator actions...
sage: asy # long time
(4/3*sqrt(3)*sqrt(r)/sqrt(pi) + 47/216*sqrt(3)/(sqrt(pi)*sqrt(r)),
 1, 4/3*sqrt(3)*sqrt(r)/sqrt(pi) + 47/216*sqrt(3)/(sqrt(pi)*sqrt(r)))
```

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```
sage: F.relative_error(asy[0], alpha, [1, 2, 4, 8], asy[1]) # long time
[((3, 3, 2), 0.9812164307, [1.515572606], [-0.54458543...]),
 ((6, 6, 4), 1.576181132, [1.992989399], [-0.26444185...]),
 ((12, 12, 8), 2.485286378, [2.712196351], [-0.091301338...]),
 ((24, 24, 16), 3.700576827, [3.760447895], [-0.016178847...])]
```

asymptotics_multiple(*p*, *alpha*, *N*, *asy_var*, *coordinate=None*, *numerical=0*, *verbose=False*)

Return the asymptotics in the given direction of a multiple point nondegenerate for `alpha`.

This is the same as *asymptotics()*, but only in the case of a convenient multiple point nondegenerate for `alpha`. Assume also that `self.dimension >= 2` and that the `p.values()` are not symbolic variables.

The formulas used for computing the asymptotic expansion are Theorem 3.4 and Theorem 3.7 of [RW2012].

INPUT:

- `p` – a dictionary with keys that can be coerced to equal `self.denominator_ring.gens()`
- `alpha` – a tuple of length `d = self.dimension()` of positive integers or, if p is a smooth point, possibly of symbolic variables
- `N` – a positive integer
- `asy_var` – (optional; default: `None`) a symbolic variable; the variable of the asymptotic expansion, if none is given, `var('r')` will be assigned
- `coordinate` – (optional; default: `None`) an integer in $\{0, \ldots, d-1\}$ indicating a convenient coordinate to base the asymptotic calculations on; if `None` is assigned, then choose `coordinate=d-1`
- `numerical` – (optional; default: 0) a natural number; if numerical is greater than 0, then return a numerical approximation of the Maclaurin ray coefficients of `self` with `numerical` digits of precision; otherwise return exact values
- `verbose` – (default: `False`) print the current state of the algorithm

OUTPUT:

The asymptotic expansion.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y,z>= PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (4 - 2*x - y - z)*(4 - x -2*y - z)
sage: Hfac = H.factor()
sage: G = 16/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(16, [(x + 2*y + z - 4, 1), (2*x + y + z - 4, 1)])
sage: p = {x: 1, y: 1, z: 1}
sage: alpha = [3, 3, 2]
sage: F.asymptotics_multiple(p, alpha, 2, var('r'), verbose=True) # long time
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second-order differential operator actions...
(4/3*sqrt(3)/(sqrt(pi)*sqrt(r)) - 25/216*sqrt(3)/(sqrt(pi)*r^(3/2)),
```

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```
 1,
 4/3*sqrt(3)/(sqrt(pi)*sqrt(r)) - 25/216*sqrt(3)/(sqrt(pi)*r^(3/2)))

sage: H = (1 - x*(1 + y))*(1 - z*x**2*(1 + 2*y))
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(1, [(x*y + x - 1, 1), (2*x^2*y*z + x^2*z - 1, 1)])
sage: p = {x: 1/2, z: 4/3, y: 1}
sage: alpha = [8, 3, 3]
sage: F.asymptotics_multiple(p, alpha, 2, var('r'), coordinate=1, verbose=True)
↪# long time
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second-order differential operator actions...
(1/172872*108^r*(24696*sqrt(7)*sqrt(3)/(sqrt(pi)*sqrt(r))
  - 1231*sqrt(7)*sqrt(3)/(sqrt(pi)*r^(3/2))),
 108,
 1/7*sqrt(7)*sqrt(3)/(sqrt(pi)*sqrt(r))
  - 1231/172872*sqrt(7)*sqrt(3)/(sqrt(pi)*r^(3/2)))
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - 2*x - y) * (1 - x - 2*y)
sage: Hfac = H.factor()
sage: G = exp(x + y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(e^(x + y), [(x + 2*y - 1, 1), (2*x + y - 1, 1)])
sage: p = {x: 1/3, y: 1/3}
sage: alpha = (var('a'), var('b'))
sage: F.asymptotics_multiple(p, alpha, 2, var('r')) # long time
(3*(1/((1/3)^a*(1/3)^b))^r*e^(2/3), 1/((1/3)^a*(1/3)^b), 3*e^(2/3))
```

asymptotics_smooth(*p*, *alpha*, *N*, *asy_var*, *coordinate=None*, *numerical=0*, *verbose=False*)

Return the asymptotics in the given direction of a smooth point.

This is the same as *asymptotics()*, but only in the case of a convenient smooth point.

The formulas used for computing the asymptotic expansions are Theorems 3.2 and 3.3 [RW2008] with the exponent of H equal to 1. Theorem 3.2 is a specialization of Theorem 3.4 of [RW2012] with $n = 1$.

INPUT:

- `p` – a dictionary with keys that can be coerced to equal `self.denominator_ring.gens()`
- `alpha` – a tuple of length `d = self.dimension()` of positive integers or, if p is a smooth point, possibly of symbolic variables
- `N` – a positive integer
- `asy_var` – (optional; default: `None`) a symbolic variable; the variable of the asymptotic expansion, if none is given, `var('r')` will be assigned

- `coordinate` – (optional; default: `None`) an integer in $\{0, \ldots, d-1\}$ indicating a convenient coordinate to base the asymptotic calculations on; if `None` is assigned, then choose `coordinate=d-1`
- `numerical` – (optional; default: 0) a natural number; if numerical is greater than 0, then return a numerical approximation of the Maclaurin ray coefficients of `self` with `numerical` digits of precision; otherwise return exact values
- `verbose` – (default: `False`) print the current state of the algorithm

OUTPUT:

The asymptotic expansion.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = 2 - 3*x
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-1/3, [(x - 2/3, 1)])
sage: alpha = [2]
sage: p = {x: 2/3}
sage: asy = F.asymptotics_smooth(p, alpha, 3, asy_var=var('r'))
sage: asy
(1/2*(9/4)^r, 9/4, 1/2)
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = 1-x-y-x*y
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = [3, 2]
sage: p = {y: 1/2*sqrt(13) - 3/2, x: 1/3*sqrt(13) - 2/3}
sage: F.asymptotics_smooth(p, alpha, 2, var('r'), numerical=3, verbose=True)
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
(71.2^r*(0.369/sqrt(r) - 0.018.../r^(3/2)), 71.2, 0.369/sqrt(r) - 0.018.../r^(3/
↪2))

sage: q = 1/2
sage: qq = q.denominator()
sage: H = 1 - q*x + q*x*y - x^2*y
sage: Hfac = H.factor()
sage: G = (1 - q*x)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = list(qq*vector([2, 1 - q]))
sage: alpha
[4, 1]
```

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```
sage: p = {x: 1, y: 1}
sage: F.asymptotics_smooth(p, alpha, 5, var('r'), verbose=True) # not tested␣
↪(140 seconds)
Creating auxiliary functions...
Computing derivatives of auxiliary functions...
Computing derivatives of more auxiliary functions...
Computing second order differential operator actions...
(1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3))
  - 1/96*sqrt(3)*2^(1/3)*gamma(2/3)/(pi*r^(5/3)),
 1,
 1/12*sqrt(3)*2^(2/3)*gamma(1/3)/(pi*r^(1/3))
  - 1/96*sqrt(3)*2^(1/3)*gamma(2/3)/(pi*r^(5/3)))
```

cohomology_decomposition()

Return the cohomology decomposition of `self`.

Let $p/(q_1^{e_1}\cdots q_n^{e_n})$ be the fraction represented by `self` and let $K[x_1,\ldots,x_d]$ be the polynomial ring in which the q_i lie. Assume that $n \leq d$ and that the gradients of the q_i are linearly independent at all points in the intersection $V_1 \cap \ldots \cap V_n$ of the algebraic varieties $V_i = \{x \in L^d \mid q_i(x) = 0\}$, where L is the algebraic closure of the field K. Return a *FractionWithFactoredDenominatorSum* f such that the differential form $f dx_1 \wedge \cdots \wedge dx_d$ is de Rham cohomologous to the differential form $p/(q_1^{e_1}\cdots q_n^{e_n}) dx_1 \wedge \cdots \wedge dx_d$ and such that the denominator of each summand of f contains no repeated irreducible factors.

The algorithm used here comes from the proof of Theorem 17.4 of [AY1983].

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x^2 + x + 1)^3
sage: decomp = FFPD(f).cohomology_decomposition()
sage: decomp
(0, []) + (2/3, [(x^2 + x + 1, 1)])
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: FFPD(1, [(x, 1), (y, 2)]).cohomology_decomposition()
(0, [])
```

The following example was fixed in trac ticket #29465:

```
sage: p = 1
sage: qs = [(x*y - 1, 1), (x**2 + y**2 - 1, 2)]
sage: f = FFPD(p, qs)
sage: f.cohomology_decomposition()
(0, []) + (-4/3*x*y, [(x^2 + y^2 - 1, 1)]) +
(1/3, [(x*y - 1, 1), (x^2 + y^2 - 1, 1)])
```

critical_cone(*p*, *coordinate=None*)

Return the critical cone of the convenient multiple point `p`.

INPUT:

- `p` – a dictionary with keys that can be coerced to equal `self.denominator_ring.gens()` and values in a field
- `coordinate` – (optional; default: `None`) a natural number

OUTPUT:

A list of vectors.

This list of vectors generate the critical cone of `p` and the cone itself, which is `None` if the values of `p` don't lie in $\mathbf{Q}$. Divide logarithmic gradients by their component `coordinate` entries. If `coordinate = None`, then search from $d-1$ down to 0 for the first index `j` such that for all `i` we have `self.log_grads()[i][j] != 0` and set `coordinate = j`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: G = 1
sage: H = (1 - x*(1 + y)) * (1 - z*x**2*(1 + 2*y))
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: p = {x: 1/2, y: 1, z: 4/3}
sage: F.critical_cone(p)
([(2, 1, 0), (3, 1, 3/2)], 2-d cone in 3-d lattice N)
```

denominator()

Return the denominator of `self`.

OUTPUT:

The denominator (i.e., the product of the factored denominator).

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.denominator()
x^3*y^2 + 2*x^3*y + x^2*y^2 + x^3 - 2*x^2*y - x*y^2 - 3*x^2 - 2*x*y
- y^2 + 3*x + 2*y - 1
```

denominator_factored()

Return the factorization in `self.denominator_ring` of the denominator of `self` but without the unit part.

OUTPUT:

The factored denominator as a list of tuple `(f, m)`, where f is a factor and m its multiplicity.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.denominator_factored()
[(x - 1, 1), (x*y + x + y - 1, 2)]
```

denominator_ring

Return the ring of the denominator.

OUTPUT:

A ring.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
sage: F = FFPD(G/H)
sage: F
(e^y, [(x - 1, 1), (x*y + x + y - 1, 2)])
sage: F.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
```

dimension()

Return the number of indeterminates of `self.denominator_ring`.

OUTPUT:

An integer.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
```

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```
sage: F.dimension()
2
```

grads(*p*)

Return a list of the gradients of the polynomials `[q for (q, e) in self.denominator_factored()]` evaluated at `p`.

INPUT:

- `p` – (optional; default: `None`) a dictionary whose keys are the generators of `self.denominator_ring`

OUTPUT:

A list.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: p = exp(x)
sage: df = [(x^3 + 3*y^2, 5), (x*y, 2), (y, 1)]
sage: f = FFPD(p, df)
sage: f
(e^x, [(y, 1), (x*y, 2), (x^3 + 3*y^2, 5)])
sage: R.gens()
(x, y)
sage: p = None
sage: f.grads(p)
[(0, 1), (y, x), (3*x^2, 6*y)]

sage: p = {x: sqrt(2), y: var('a')}
sage: f.grads(p)
[(0, 1), (a, sqrt(2)), (6, 6*a)]
```

is_convenient_multiple_point(*p*)

Tests if `p` is a convenient multiple point of `self`.

In case `p` is a convenient multiple point, `verdict = True` and `comment` is a string stating which variables it's convenient to use. In case `p` is not, `verdict = False` and `comment` is a string explaining why `p` fails to be a convenient multiple point.

See [RW2012] for more details.

INPUT:

- `p` – a dictionary with keys that can be coerced to equal `self.denominator_ring.gens()`

OUTPUT:

A pair `(verdict, comment)`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y,z> = PolynomialRing(QQ)
```

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```
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (1 - x*(1 + y)) * (1 - z*x**2*(1 + 2*y))
sage: df = H.factor()
sage: G = 1 / df.unit()
sage: F = FFPD(G, df)
sage: p1 = {x: 1/2, y: 1, z: 4/3}
sage: p2 = {x: 1, y: 2, z: 1/2}
sage: F.is_convenient_multiple_point(p1)
(True, 'convenient in variables [x, y]')
sage: F.is_convenient_multiple_point(p2)
(False, 'not a singular point')
```

leinartas_decomposition()

Return a Leinartas decomposition of `self`.

Let $f = p/q$ where q lies in a d -variate polynomial ring $K[X]$ for some field K. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in $K[X]$ into irreducible factors and let V_i be the algebraic variety $\{x \in L^d \mid q_i(x) = 0\}$ of q_i over the algebraic closure L of K. By [Rai2012], f can be written as

$$(*) \quad \sum_A \frac{p_A}{\prod_{i \in A} q_i^{b_i}},$$

where the b_i are positive integers, each p_A is a product of p and an element of $K[X]$, and the sum is taken over all subsets $A \subseteq \{1, \ldots, m\}$ such that

1. $|A| \leq d$,
2. $\bigcap_{i \in A} T_i \neq \emptyset$, and
3. $\{q_i \mid i \in A\}$ is algebraically independent.

In particular, any rational expression in d variables can be represented as a sum of rational expressions whose denominators each contain at most d distinct irreducible factors.

We call $(*)$ a *Leinartas decomposition* of f. Leinartas decompositions are not unique.

The algorithm used comes from [Rai2012].

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = (x^2 + 1)/((x + 2)*(x - 1)*(x^2 + x + 1))
sage: decomp = FFPD(f).leinartas_decomposition()
sage: decomp
(0, []) + (2/9, [(x - 1, 1)]) +
(-5/9, [(x + 2, 1)]) + (1/3*x, [(x^2 + x + 1, 1)])
sage: decomp.sum().quotient() == f
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/x + 1/y + 1/(x*y + 1)
sage: decomp = FFPD(f).leinartas_decomposition()
sage: decomp
(0, []) + (1, [(x*y + 1, 1)]) + (x + y, [(y, 1), (x, 1)])
sage: decomp.sum().quotient() == f
True
sage: def check_decomp(r):
....:     L = r.nullstellensatz_certificate()
....:     J = r.algebraic_dependence_certificate()
....:     return L is None and (J is None or J == J.ring().ideal())
sage: all(check_decomp(r) for r in decomp)
True
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = sin(x)/x + 1/y + 1/(x*y + 1)
sage: G = f.numerator()
sage: H = R(f.denominator())
sage: ff = FFPD(G, H.factor())
sage: decomp = ff.leinartas_decomposition()
sage: decomp
(0, []) +
(-(x*y^2*sin(x) + x^2*y + x*y + y*sin(x) + x)*y, [(y, 1)]) +
((x*y^2*sin(x) + x^2*y + x*y + y*sin(x) + x)*x*y, [(x*y + 1, 1)]) +
(x*y^2*sin(x) + x^2*y + x*y + y*sin(x) + x, [(y, 1), (x, 1)])
sage: bool(decomp.sum().quotient() == f)
True
sage: all(check_decomp(r) for r in decomp)
True
```

```
sage: R.<x,y,z>= PolynomialRing(GF(2, 'a'))
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x * y * z * (x*y + z))
sage: decomp = FFPD(f).leinartas_decomposition()
sage: decomp
(0, []) + (1, [(z, 2), (x*y + z, 1)]) +
(1, [(z, 2), (y, 1), (x, 1)])
sage: decomp.sum().quotient() == f
True
```

log_grads(p)

Return a list of the logarithmic gradients of the polynomials `[q for (q, e) in self.denominator_factored()]` evaluated at `p`.

The logarithmic gradient of a function f at point p is the vector $(x_1\partial_1 f(x), \ldots, x_d\partial_d f(x))$ evaluated at p.

INPUT:

- `p` – (optional; default: `None`) a dictionary whose keys are the generators of `self.denominator_ring`

OUTPUT:

A list.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: p = exp(x)
sage: df = [(x^3 + 3*y^2, 5), (x*y, 2), (y, 1)]
sage: f = FFPD(p, df)
sage: f
(e^x, [(y, 1), (x*y, 2), (x^3 + 3*y^2, 5)])
sage: R.gens()
(x, y)
sage: p = None
sage: f.log_grads(p)
[(0, y), (x*y, x*y), (3*x^3, 6*y^2)]

sage: p = {x: sqrt(2), y: var('a')}
sage: f.log_grads(p)
[(0, a), (sqrt(2)*a, sqrt(2)*a), (6*sqrt(2), 6*a^2)]
```

maclaurin_coefficients(*multi_indices*, *numerical=0*)

Return the Maclaurin coefficients of `self` with given `multi_indices`.

INPUT:

- `multi_indices` – a list of tuples of positive integers, where each tuple has length `self.dimension()`
- `numerical` – (optional; default: 0) a natural number; if positive, return numerical approximations of coefficients with `numerical` digits of accuracy

OUTPUT:

A dictionary whose value of the key `nu` are the Maclaurin coefficient of index `nu` of `self`.

Note: Uses iterated univariate Maclaurin expansions. Slow.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = 2 - 3*x
sage: Hfac = H.factor()
sage: G = 1 / Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-1/3, [(x - 2/3, 1)])
sage: F.maclaurin_coefficients([(2*k,) for k in range(6)])
{(0,): 1/2,
 (2,): 9/8,
 (4,): 81/32,
 (6,): 729/128,
```

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```
 (8,): 6561/512,
 (10,): 59049/2048}
```

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (4 - 2*x - y - z) * (4 - x - 2*y - z)
sage: Hfac = H.factor()
sage: G = 16 / Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = vector([3, 3, 2])
sage: interval = [1, 2, 4]
sage: S = [r*alpha for r in interval]
sage: F.maclaurin_coefficients(S, numerical=10)
{(3, 3, 2): 0.7849731445,
 (6, 6, 4): 0.7005249476,
 (12, 12, 8): 0.5847732654}
```

nullstellensatz_certificate()

Return a Nullstellensatz certificate of `self` if it exists.

Let $[(q_1, e_1), \ldots, (q_n, e_n)]$ be the denominator factorization of `self`. The Nullstellensatz certificate is a list of polynomials $h_1, \ldots, h_m$ in `self.denominator_ring` that satisfies $h_1 q_1 + \cdots + h_m q_n = 1$ if it exists.

Note: Only works for multivariate base rings.

OUTPUT:

A list of polynomials or `None` if no Nullstellensatz certificate exists.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = sin(x)
sage: H = x^2 * (x*y + 1)
sage: f = FFPD(G, H.factor())
sage: L = f.nullstellensatz_certificate()
sage: L
[y^2, -x*y + 1]
sage: df = f.denominator_factored()
sage: sum(L[i]*df[i][0]**df[i][1] for i in range(len(df))) == 1
True
```

```
sage: f = 1/(x*y)
sage: L = FFPD(f).nullstellensatz_certificate()
sage: L is None
True
```

nullstellensatz_decomposition()

Return a Nullstellensatz decomposition of `self`.

Let $f = p/q$ where q lies in a d-variate polynomial ring $K[X]$ for some field K and $d \geq 1$. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in $K[X]$ into irreducible factors and let V_i be the algebraic variety $\{x \in L^d \mid q_i(x) = 0\}$ of q_i over the algebraic closure L of K. By [Rai2012], f can be written as

$$(*) \quad \sum_A \frac{p_A}{\prod_{i \in A} q_i^{e_i}},$$

where the p_A are products of p and elements in $K[X]$ and the sum is taken over all subsets $A \subseteq \{1, \ldots, m\}$ such that $\bigcap_{i \in A} T_i \neq \emptyset$.

We call $(*)$ a *Nullstellensatz decomposition* of f. Nullstellensatz decompositions are not unique.

The algorithm used comes from [Rai2012].

Note: Recursive. Only works for multivariate `self`.

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import *
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 1/(x*(x*y + 1))
sage: decomp = FFPD(f).nullstellensatz_decomposition()
sage: decomp
(0, []) + (1, [(x, 1)]) + (-y, [(x*y + 1, 1)])
sage: decomp.sum().quotient() == f
True
sage: [r.nullstellensatz_certificate() is None for r in decomp]
[True, True, True]
```

```
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: G = sin(y)
sage: H = x*(x*y + 1)
sage: f = FFPD(G, H.factor())
sage: decomp = f.nullstellensatz_decomposition()
sage: decomp
(0, []) + (sin(y), [(x, 1)]) + (-y*sin(y), [(x*y + 1, 1)])
sage: bool(decomp.sum().quotient() == G/H)
True
sage: [r.nullstellensatz_certificate() is None for r in decomp]
[True, True, True]
```

numerator()

Return the numerator of `self`.

OUTPUT:

The numerator.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.numerator()
-e^y
```

numerator_ring

Return the ring of the numerator.

OUTPUT:

A ring.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F.numerator_ring
Symbolic Ring
sage: F = FFPD(G/H)
sage: F
(e^y, [(x - 1, 1), (x*y + x + y - 1, 2)])
sage: F.numerator_ring
Symbolic Ring
```

quotient()

Convert `self` into a quotient.

OUTPUT:

An element.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: H = (1 - x - y - x*y)**2*(1-x)
sage: Hfac = H.factor()
sage: G = exp(y)/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: F
(-e^y, [(x - 1, 1), (x*y + x + y - 1, 2)])
sage: F.quotient()
```

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```
-e^y/(x^3*y^2 + 2*x^3*y + x^2*y^2 + x^3 - 2*x^2*y - x*y^2 - 3*x^2 -
2*x*y - y^2 + 3*x + 2*y - 1)
```

relative_error(*approx*, *alpha*, *interval*, *exp_scale=1*, *digits=10*)

Return the relative error between the values of the Maclaurin coefficients of `self` with multi-indices `r` `alpha` for `r` in `interval` and the values of the functions (of the variable `r`) in `approx`.

INPUT:

- `approx` – an individual or list of symbolic expressions in one variable
- `alpha` - a list of positive integers of length `self.denominator_ring.ngens()`
- `interval` – a list of positive integers
- `exp_scale` – (optional; default: 1) a number

OUTPUT:

A list of tuples with properties described below.

This outputs a list whose entries are a tuple `(r*alpha, a_r, b_r, err_r)` for `r` in `interval`. Here `r*alpha` is a tuple; `a_r` is the `r*alpha` (multi-index) coefficient of the Maclaurin series for `self` divided by `exp_scale**r`; `b_r` is a list of the values of the functions in `approx` evaluated at `r` and divided by `exp_scale**m`; `err_r` is the list of relative errors `(a_r - f)/a_r` for `f` in `b_r`. All outputs are decimal approximations.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = 1 - x - y - x*y
sage: Hfac = H.factor()
sage: G = 1 / Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = [1, 1]
sage: r = var('r')
sage: a1 = (0.573/sqrt(r))*5.83^r
sage: a2 = (0.573/sqrt(r) - 0.0674/r^(3/2))*5.83^r
sage: es = 5.83
sage: F.relative_error([a1, a2], alpha, [1, 2, 4, 8], es) # long time
[((1, 1), 0.5145797599,
  [0.5730000000, 0.5056000000], [-0.1135300000, 0.01745066667]),
 ((2, 2), 0.3824778089,
  [0.4051721856, 0.3813426871], [-0.05933514614, 0.002967810973]),
 ((4, 4), 0.2778630595,
  [0.2865000000, 0.2780750000], [-0.03108344267, -0.0007627515584]),
 ((8, 8), 0.1991088276,
  [0.2025860928, 0.1996074055], [-0.01746414394, -0.002504047242])]
```

singular_ideal()

Return the singular ideal of `self`.

Let R be the ring of `self` and H its denominator. Let H_{red} be the reduction (square-free part) of H. Return the ideal in R generated by H_{red} and its partial derivatives. If the coefficient field of R is algebraically closed, then the output is the ideal of the singular locus (which is a variety) of the variety of H.

OUTPUT:

An ideal.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (1 - x*(1 + y))^3 * (1 - z*x**2*(1 + 2*y))
sage: df = H.factor()
sage: G = 1 / df.unit()
sage: F = FFPD(G, df)
sage: F.singular_ideal()
Ideal (x*y + x - 1, y^2 - 2*y*z + 2*y - z + 1, x*z + y - 2*z + 1) of
 Multivariate Polynomial Ring in x, y, z over Rational Field
```

smooth_critical_ideal(*alpha*)

Return the smooth critical ideal of `self`.

Let R be the ring of `self` and H its denominator. Return the ideal in R of smooth critical points of the variety of H for the direction `alpha`. If the variety V of H has no smooth points, then return the ideal in R of V.

See [RW2012] for more details.

INPUT:

- `alpha` – a tuple of positive integers and/or symbolic entries of length `self.denominator_ring.ngens()`

OUTPUT:

An ideal.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: H = (1 - x - y - x*y)^2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = var('a1, a2')
sage: F.smooth_critical_ideal(alpha)
Ideal (y^2 + (2*a1)/a2*y - 1, x + (-a2)/a1*y + (-a1 + a2)/a1) of
 Multivariate Polynomial Ring in x, y over Fraction Field of
 Multivariate Polynomial Ring in a1, a2 over Rational Field

sage: H = (1-x-y-x*y)^2
sage: Hfac = H.factor()
sage: G = 1/Hfac.unit()
sage: F = FFPD(G, Hfac)
sage: alpha = [7/3, var('a')]
```

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```
sage: F.smooth_critical_ideal(alpha)
Ideal (y^2 + 14/(3*a)*y - 1, x + (-3*a)/7*y + (3*a - 7)/7) of Multivariate␣
↪Polynomial Ring in x, y over Fraction Field of Univariate Polynomial Ring in␣
↪a over Rational Field
```

univariate_decomposition()

Return the usual univariate partial fraction decomposition of `self`.

Assume that the numerator of `self` lies in the same univariate factorial polynomial ring as the factors of the denominator.

Let $f = p/q$ be a rational expression where p and q lie in a univariate factorial polynomial ring R. Let $q_1^{e_1} \cdots q_n^{e_n}$ be the unique factorization of q in R into irreducible factors. Then f can be written uniquely as:

$$(*) \quad p_0 + \sum_{i=1}^{m} \frac{p_i}{q_i^{e_i}},$$

for some $p_j \in R$. We call $(*)$ the *usual partial fraction decomposition* of f.

Note: This partial fraction decomposition can be computed using `partial_fraction()` or `partial_fraction_decomposition()` as well. However, here we use the already obtained/cached factorization of the denominator. This gives a speed up for non-small instances.

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
```

One variable:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: f
(5*x^7 - 5*x^6 + 5/3*x^5 - 5/3*x^4 + 2*x^3 - 2/3*x^2 + 1/3*x - 1/3)/(x^4 - x^3␣
↪+ 1/3*x^2 - 1/3*x)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(5*x^3, []) +
(1, [(x - 1, 1)]) +
(1, [(x, 1)]) +
(1/3, [(x^2 + 1/3, 1)])
sage: decomp.sum().quotient() == f
True
```

One variable with numerator in symbolic ring:

```
sage: R.<x> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = 5*x^3 + 1/x + 1/(x-1) + exp(x)/(3*x^2 + 1)
```

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```
sage: f
(5*x^5 - 5*x^4 + 2*x - 1)/(x^2 - x) + e^x/(3*x^2 + 1)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(0, []) +
(15/4*x^7 - 15/4*x^6 + 5/4*x^5 - 5/4*x^4 + 3/2*x^3 + 1/4*x^2*e^x -
 3/4*x^2 - 1/4*x*e^x + 1/2*x - 1/4, [(x - 1, 1)]) +
(-15*x^7 + 15*x^6 - 5*x^5 + 5*x^4 - 6*x^3 -
 x^2*e^x + 3*x^2 + x*e^x - 2*x + 1, [(x, 1)]) +
(1/4*(15*x^7 - 15*x^6 + 5*x^5 - 5*x^4 + 6*x^3 + x^2*e^x -
      3*x^2 - x*e^x + 2*x - 1)*(3*x - 1), [(x^2 + 1/3, 1)])
```

One variable over a finite field:

```
sage: R.<x> = PolynomialRing(GF(2))
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: f
(x^6 + x^4 + 1)/(x^3 + x)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(x^3, []) + (1, [(x, 1)]) + (x, [(x + 1, 2)])
sage: decomp.sum().quotient() == f
True
```

One variable over an inexact field:

```
sage: R.<x> = PolynomialRing(CC)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = 5*x^3 + 1/x + 1/(x-1) + 1/(3*x^2 + 1)
sage: f
(5.00000000000000*x^7 - 5.00000000000000*x^6 + 1.66666666666667*x^5 - 1.
↪66666666666667*x^4 + 2.00000000000000*x^3 - 0.666666666666667*x^2 + 0.
↪333333333333333*x - 0.333333333333333)/(x^4 - x^3 + 0.333333333333333*x^2 - 0.
↪333333333333333*x)
sage: decomp = FFPD(f).univariate_decomposition()
sage: decomp
(5.00000000000000*x^3, []) +
(1.00000000000000, [(x - 1.00000000000000, 1)]) +
(-0.288675134594813*I, [(x - 0.577350269189626*I, 1)]) +
(1.00000000000000, [(x, 1)]) +
(0.288675134594813*I, [(x + 0.577350269189626*I, 1)])
sage: decomp.sum().quotient() == f # Rounding error coming
False
```

AUTHORS:

- Robert Bradshaw (2007-05-31)
- Alexander Raichev (2012-06-25)
- Daniel Krenn (2014-12-01)

class sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**FractionWithFactoredDenominat**

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.rings.ring.Ring

This is the ring of fractions with factored denominator.

INPUT:

- `denominator_ring` – the base ring (a polynomial ring)
- `numerator_ring` – (optional) the numerator ring; the default is the `denominator_ring`
- `category` – (default: `Rings`) the category

See also:

FractionWithFactoredDenominator, *asymptotics_multivariate_generating_functions*

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: df = [x, 1], [y, 1], [x*y+1, 1]
sage: f = FFPD(x, df)  # indirect doctest
sage: f
(1, [(y, 1), (x*y + 1, 1)])
```

AUTHORS:

- Daniel Krenn (2014-12-01)

Element
alias of *FractionWithFactoredDenominator*

base_ring()
Returns the base ring.

OUTPUT:

A ring.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing
sage: P.<X, Y> = ZZ[]
sage: F = FractionWithFactoredDenominatorRing(P); F
Ring of fractions with factored denominator
over Multivariate Polynomial Ring in X, Y over Integer Ring
sage: F.base_ring()
Integer Ring
sage: F.base()
Multivariate Polynomial Ring in X, Y over Integer Ring
```

class sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominatorSum

Bases: list

A list representing the sum of *FractionWithFactoredDenominator* objects with distinct denominator factorizations.

AUTHORS:

- Alexander Raichev (2012-06-25)
- Daniel Krenn (2014-12-01)

denominator_ring

Return the polynomial ring of the denominators of self.

OUTPUT:

A ring or None if the list is empty.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing, FractionWithFactoredDenominatorSum
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = FFPD(x + y, [(y, 1), (x, 1)])
sage: s = FractionWithFactoredDenominatorSum([f])
sage: s.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
sage: g = FFPD(x + y, [])
sage: t = FractionWithFactoredDenominatorSum([g])
sage: t.denominator_ring
Multivariate Polynomial Ring in x, y over Rational Field
```

sum()

Return the sum of the elements in self.

OUTPUT:

An instance of *FractionWithFactoredDenominator*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing, FractionWithFactoredDenominatorSum
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: df = (x, 1), (y, 1), (x*y + 1, 1)
sage: f = FFPD(2, df)
sage: g = FFPD(2*x*y, df)
sage: FractionWithFactoredDenominatorSum([f, g])
(2, [(y, 1), (x, 1), (x*y + 1, 1)]) + (2, [(x*y + 1, 1)])
sage: FractionWithFactoredDenominatorSum([f, g]).sum()
(2, [(y, 1), (x, 1)])

sage: f = FFPD(cos(x), [(x, 2)])
sage: g = FFPD(cos(y), [(x, 1), (y, 2)])
sage: FractionWithFactoredDenominatorSum([f, g])
```

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```
(cos(x), [(x, 2)]) + (cos(y), [(y, 2), (x, 1)])
sage: FractionWithFactoredDenominatorSum([f, g]).sum()
(y^2*cos(x) + x*cos(y), [(y, 2), (x, 2)])
```

whole_and_parts()

Rewrite `self` as a sum of a (possibly zero) polynomial followed by reduced rational expressions.

OUTPUT:

An instance of *FractionWithFactoredDenominatorSum*.

Only useful for multivariate decompositions.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing, FractionWithFactoredDenominatorSum
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R, SR)
sage: f = x**2 + 3*y + 1/x + 1/y
sage: f = FFPD(f); f
(x^3*y + 3*x*y^2 + x + y, [(y, 1), (x, 1)])
sage: FractionWithFactoredDenominatorSum([f]).whole_and_parts()
(x^2 + 3*y, []) + (x + y, [(y, 1), (x, 1)])

sage: f = cos(x)**2 + 3*y + 1/x + 1/y; f
cos(x)^2 + 3*y + 1/x + 1/y
sage: G = f.numerator()
sage: H = R(f.denominator())
sage: f = FFPD(G, H.factor()); f
(x*y*cos(x)^2 + 3*x*y^2 + x + y, [(y, 1), (x, 1)])
sage: FractionWithFactoredDenominatorSum([f]).whole_and_parts()
(0, []) + (x*y*cos(x)^2 + 3*x*y^2 + x + y, [(y, 1), (x, 1)])
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**coerce_point**(*R*, *p*)

Coerce the keys of the dictionary `p` into the ring `R`.

Warning: This method assumes that it is possible.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import FractionWithFactoredDenominatorRing, coerce_point
sage: R.<x,y> = PolynomialRing(QQ)
sage: FFPD = FractionWithFactoredDenominatorRing(R)
sage: f = FFPD()
sage: p = {SR(x): 1, SR(y): 7/8}
sage: for k in sorted(p, key=str):
....:     print("{} {} {}".format(k, k.parent(), p[k]))
x Symbolic Ring 1
y Symbolic Ring 7/8
sage: q = coerce_point(R, p)
sage: for k in sorted(q, key=str):
```

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```
....:     print("{} {} {}".format(k, k.parent(), q[k]))
x Multivariate Polynomial Ring in x, y over Rational Field 1
y Multivariate Polynomial Ring in x, y over Rational Field 7/8
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**diff_all**(*f*, *V*, *n*, *ending=[]*, *sub=None*, *sub_final=None*, *zero_order=0*, *rekey=None*)

Return a dictionary of representative mixed partial derivatives of f from order 1 up to order n with respect to the variables in V.

The default is to key the dictionary by all nondecreasing sequences in V of length 1 up to length n.

INPUT:

- `f` – an individual or list of $\mathcal{C}^{n+1}$ functions
- `V` – a list of variables occurring in f
- `n` – a natural number
- `ending` – a list of variables in V
- `sub` – an individual or list of dictionaries
- `sub_final` – an individual or list of dictionaries
- `rekey` – a callable symbolic function in V or list thereof
- `zero_order` – a natural number

OUTPUT:

The dictionary `{s_1:deriv_1, ..., sr:deriv_r}`.

Here `s_1, ..., s_r` is a listing of all nondecreasing sequences of length 1 up to length n over the alphabet V, where $w > v$ in X if and only if `str(w) > str(v)`, and `deriv_j` is the derivative of f with respect to the derivative sequence `s_j` and simplified with respect to the substitutions in `sub` and evaluated at `sub_final`. Moreover, all derivatives with respect to sequences of length less than `zero_order` (derivatives of order less than `zero_order`) will be made zero.

If `rekey` is nonempty, then `s_1, ..., s_r` will be replaced by the symbolic derivatives of the functions in `rekey`.

If `ending` is nonempty, then every derivative sequence `s_j` will be suffixed by `ending`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import diff_all
sage: f = function('f')(x)
sage: dd = diff_all(f, [x], 3)
sage: dd[(x, x, x)]
diff(f(x), x, x, x)

sage: d1 = {diff(f, x): 4*x^3}
sage: dd = diff_all(f, [x], 3, sub=d1)
sage: dd[(x, x, x)]
```

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```
24*x

sage: dd = diff_all(f, [x], 3, sub=d1, rekey=f)
sage: dd[diff(f, x, 3)]
24*x

sage: a = {x:1}
sage: dd = diff_all(f, [x], 3, sub=d1, rekey=f, sub_final=a)
sage: dd[diff(f, x, 3)]
24
```

```
sage: X = var('x, y, z')
sage: f = function('f')(*X)
sage: dd = diff_all(f, X, 2, ending=[y, y, y])
sage: dd[(z, y, y, y)]
diff(f(x, y, z), y, y, y, z)
```

```
sage: g = function('g')(*X)
sage: dd = diff_all([f, g], X, 2)
sage: dd[(0, y, z)]
diff(f(x, y, z), y, z)

sage: dd[(1, z, z)]
diff(g(x, y, z), z, z)

sage: f = exp(x*y*z)
sage: ff = function('ff')(*X)
sage: dd = diff_all(f, X, 2, rekey=ff)
sage: dd[diff(ff, x, z)]
x*y^2*z*e^(x*y*z) + y*e^(x*y*z)
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**diff_op**(*A*, *B*, *AB_derivs*, *V*, *M*, *r*, *N*)

Return the derivatives $DD^{(l+k)}(A[j]B^l)$ evaluated at a point p for various natural numbers j, k, l which depend on r and N.

Here DD is a specific second-order linear differential operator that depends on M , A is a list of symbolic functions, B is symbolic function, and `AB_derivs` contains all the derivatives of A and B evaluated at p that are necessary for the computation.

INPUT:

- `A` – a single or length `r` list of symbolic functions in the variables `V`
- `B` – a symbolic function in the variables `V`.
- `AB_derivs` – a dictionary whose keys are the (symbolic) derivatives of `A[0], ..., A[r-1]` up to order `2 * N-2` and the (symbolic) derivatives of `B` up to order `2 * N`; the values of the dictionary are complex numbers that are the keys evaluated at a common point p
- `V` – the variables of the `A[j]` and `B`
- `M` – a symmetric $l \times l$ matrix, where l is the length of `V`
- `r, N` – natural numbers

OUTPUT:

A dictionary.

The output is a dictionary whose keys are natural number tuples of the form (j, k, l), where $l \leq 2k$, $j \leq r-1$, and $j+k \leq N-1$, and whose values are $DD^{(}l+k)(A[j]B^{l})$ evaluated at a point p, where DD is the linear second-order differential operator $-\sum_{i=0}^{l-1}\sum_{j=0}^{l-1} M[i][j]\partial^2/(\partial V[j]\partial V[i])$.

Note: For internal use by *FractionWithFactoredDenominator.asymptotics_smooth()* and *FractionWithFactoredDenominator.asymptotics_multiple()*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import diff_op
sage: T = var('x, y')
sage: A = function('A')(*tuple(T))
sage: B = function('B')(*tuple(T))
sage: AB_derivs = {}
sage: M = matrix([[1, 2],[2, 1]])
sage: DD = diff_op(A, B, AB_derivs, T, M, 1, 2)
sage: sorted(DD)
[(0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 1, 2)]
sage: DD[(0, 1, 2)].number_of_operands()
246
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**diff_op_simple**(*A*, *B*, *AB_derivs*, *x*, *v*, *a*, *N*)

Return $DD^{(}ek+vl)(AB^{l})$ evaluated at a point p for various natural numbers e, k, l that depend on v and N.

Here DD is a specific linear differential operator that depends on a and v , A and B are symbolic functions, and AB_derivs contains all the derivatives of A and B evaluated at p that are necessary for the computation.

Note: For internal use by the function *FractionWithFactoredDenominator.asymptotics_smooth()*.

INPUT:

- `A, B` – Symbolic functions in the variable `x`
- `AB_derivs` - a dictionary whose keys are the (symbolic) derivatives of `A` up to order `2 * N` if `v` is even or `N` if `v` is odd and the (symbolic) derivatives of `B` up to order `2 * N + v` if `v` is even or `N + v` if `v` is odd; the values of the dictionary are complex numbers that are the keys evaluated at a common point p
- `x` – a symbolic variable
- `a` – a complex number
- `v, N` – natural numbers

OUTPUT:

A dictionary.

The output is a dictionary whose keys are natural number pairs of the form (k, l), where $k < N$ and $l \leq 2k$ and whose values are $DD^{(}ek+vl)(AB^{l})$ evaluated at a point p. Here $e = 2$ if v is even, $e = 1$ if v is odd, and DD

is the linear differential operator $(a^{-1/v}d/dt)$ if v is even and $(|a|^{-1/v}i\mathrm{sgn}(a)d/dt)$ if v is odd.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import diff_op_simple
sage: A = function('A')(x)
sage: B = function('B')(x)
sage: AB_derivs = {}
sage: sorted(diff_op_simple(A, B, AB_derivs, x, 3, 2, 2).items())
[((0, 0), A(x)),
 ((1, 0), 1/2*I*2^(2/3)*diff(A(x), x)),
 ((1, 1),
  1/4*2^(2/3)*(B(x)*diff(A(x), x, x, x, x) + 4*diff(A(x), x, x, x)*diff(B(x), x) +␣
↪6*diff(A(x), x, x)*diff(B(x), x, x) + 4*diff(A(x), x)*diff(B(x), x, x, x) +␣
↪A(x)*diff(B(x), x, x, x, x)))]
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**diff_prod**(*f_derivs*, *u*, *g*, *X*, *interval*, *end*, *uderivs*, *atc*)

Take various derivatives of the equation $f = ug$, evaluate them at a point c, and solve for the derivatives of u.

INPUT:

- `f_derivs` – a dictionary whose keys are all tuples of the form `s + end`, where `s` is a sequence of variables from `X` whose length lies in `interval`, and whose values are the derivatives of a function f evaluated at c
- `u` – a callable symbolic function
- `g` – an expression or callable symbolic function
- `X` – a list of symbolic variables
- `interval` – a list of positive integers Call the first and last values n and nn, respectively
- `end` – a possibly empty list of repetitions of the variable `z`, where `z` is the last element of `X`
- `uderivs` – a dictionary whose keys are the symbolic derivatives of order 0 to order $n-1$ of `u` evaluated at c and whose values are the corresponding derivatives evaluated at c
- `atc` – a dictionary whose keys are the keys of c and all the symbolic derivatives of order 0 to order nn of `g` evaluated c and whose values are the corresponding derivatives evaluated at c

OUTPUT:

A dictionary whose keys are the derivatives of `u` up to order nn and whose values are those derivatives evaluated at c.

This function works by differentiating the equation $f = ug$ with respect to the variable sequence `s + end`, for all tuples `s` of `X` of lengths in `interval`, evaluating at the point c , and solving for the remaining derivatives of `u`. This function assumes that `u` never appears in the differentiations of $f = ug$ after evaluating at c.

Note: For internal use by *FractionWithFactoredDenominator.asymptotics_multiple()*.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import diff_prod
sage: u = function('u')(x)
sage: g = function('g')(x)
sage: fd = {(x,):1,(x, x):1}
sage: ud = {u(x=2): 1}
sage: atc = {x: 2, g(x=2): 3, diff(g, x)(x=2): 5}
sage: atc[diff(g, x, x)(x=2)] = 7
sage: dd = diff_prod(fd, u, g, [x], [1, 2], [], ud, atc)
sage: dd[diff(u, x, 2)(x=2)]
22/9
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**diff_seq**(*V*, *s*)

Given a list `s` of tuples of natural numbers, return the list of elements of `V` with indices the elements of the elements of `s`.

INPUT:

- `V` – a list
- `s` – a list of tuples of natural numbers in the interval `range(len(V))`

OUTPUT:

The tuple `tuple([V[tt] for tt in sorted(t)])`, where `t` is the list of elements of the elements of `s`.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import diff_seq
sage: V = list(var('x, t, z'))
sage: diff_seq(V,([0, 1],[0, 2, 1],[0, 0]))
(x, x, x, x, t, t, z)
```

Note: This function is for internal use by *diff_op()*.

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**direction**(*v*, *coordinate=None*)

Return `[vv/v[coordinate] for vv in v]` where `coordinate` is the last index of `v` if not specified otherwise.

INPUT:

- `v` – a vector
- `coordinate` – (optional; default: `None`) an index for `v`

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import direction
sage: direction([2, 3, 5])
(2/5, 3/5, 1)
sage: direction([2, 3, 5], 0)
(1, 3/2, 5/2)
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**permutation_sign**(*s*, *u*)
This function returns the sign of the permutation on `1, ..., len(u)` that is induced by the sublist `s` of `u`.

Note: This function was intended for internal use and is deprecated now (trac ticket #29465).

INPUT:

- `s` – a sublist of `u`
- `u` – a list

OUTPUT:

The sign of the permutation obtained by taking indices within `u` of the list `s + sc`, where `sc` is `u` with the elements of `s` removed.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import permutation_sign
sage: u = ['a', 'b', 'c', 'd', 'e']
sage: s = ['b', 'd']
sage: permutation_sign(s, u)
doctest:...: DeprecationWarning: the function permutation_sign is deprecated
See https://trac.sagemath.org/29465 for details.
-1
sage: s = ['d', 'b']
sage: permutation_sign(s, u)
1
```

sage.rings.asymptotic.asymptotics_multivariate_generating_functions.**subs_all**(*f*, *sub*, *simplify=False*)
Return the items of *f* substituted by the dictionaries of `sub` in order of their appearance in `sub`.

INPUT:

- `f` – an individual or list of symbolic expressions or dictionaries
- `sub` – an individual or list of dictionaries
- `simplify` – (default: `False`) boolean; set to `True` to simplify the result

OUTPUT:

The items of `f` substituted by the dictionaries of `sub` in order of their appearance in `sub`. The `subs()` command is used. If simplify is `True`, then `simplify()` is used after substitution.

EXAMPLES:

```
sage: from sage.rings.asymptotic.asymptotics_multivariate_generating_functions␣
↪import subs_all
sage: var('x, y, z')
(x, y, z)
sage: a = {x:1}
sage: b = {y:2}
sage: c = {z:3}
sage: subs_all(x + y + z, a)
y + z + 1
```

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```
sage: subs_all(x + y + z, [c, a])
y + 4
sage: subs_all([x + y + z, y^2], b)
[x + z + 2, 4]
sage: subs_all([x + y + z, y^2], [b, c])
[x + 5, 4]
```

```
sage: var('x, y')
(x, y)
sage: a = {'foo': x**2 + y**2, 'bar': x - y}
sage: b = {x: 1 , y: 2}
sage: subs_all(a, b)
{'bar': -1, 'foo': 5}
```

CHAPTER

FIVE

INDICES AND TABLES

- Index
- Module Index
- Search Page

PYTHON MODULE INDEX

r

sage.rings.asymptotic.asymptotic_expansion_generators, 39
sage.rings.asymptotic.asymptotic_ring, 7
sage.rings.asymptotic.asymptotics_multivariate_generating_functions, 120
sage.rings.asymptotic.growth_group, 48
sage.rings.asymptotic.growth_group_cartesian, 74
sage.rings.asymptotic.misc, 112
sage.rings.asymptotic.term_monoid, 82

INDEX

A

absorb() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 94
absorption() (*in module sage.rings.asymptotic.term_monoid*), 111
AbstractGrowthGroupFunctor (*class in sage.rings.asymptotic.growth_group*), 52
AdditiveMagmas (*sage.rings.asymptotic.growth_group.GenericGrowthGroup attribute*), 61
AdditiveMagmas (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup attribute*), 68
algebraic_dependence_certificate() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 125
algebraic_dependence_decomposition() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 126
asymptotic_decomposition() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 127
asymptotic_expansions (*in module sage.rings.asymptotic.asymptotic_expansion_generators*), 48
AsymptoticExpansion (*class in sage.rings.asymptotic.asymptotic_ring*), 13
AsymptoticExpansionGenerators (*class in sage.rings.asymptotic.asymptotic_expansion_generators*), 40
AsymptoticRing (*class in sage.rings.asymptotic.asymptotic_ring*), 30
AsymptoticRingFunctor (*class in sage.rings.asymptotic.asymptotic_ring*), 38
asymptotics() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 128
asymptotics_multiple() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 130
asymptotics_smooth() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 131

B

B() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 15
B() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing static method*), 32
base (*sage.rings.asymptotic.growth_group.ExponentialGrowthElement attribute*), 53
base (*sage.rings.asymptotic.growth_group.GrowthGroupFactor attribute*), 66
base_ring() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 147
bidirectional_merge_overlapping() (*in module sage.rings.asymptotic.misc*), 114
bidirectional_merge_sorted() (*in module sage.rings.asymptotic.misc*), 114
Binomial_kn_over_En() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 40
BTerm (*class in sage.rings.asymptotic.term_monoid*), 85
BTermMonoid (*class in sage.rings.asymptotic.term_monoid*), 87

C

can_absorb() (*in module sage.rings.asymptotic.term_monoid*), 112
can_absorb() (*sage.rings.asymptotic.term_monoid.BTerm method*), 86
can_absorb() (*sage.rings.asymptotic.term_monoid.ExactTerm method*), 89
can_absorb() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 95
can_absorb() (*sage.rings.asymptotic.term_monoid.OTerm method*), 104
cartesian_injection() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 80
CartesianProductFactory (*class in sage.rings.asymptotic.growth_group_cartesian*), 74
change_parameter() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 32

change_parameter() (*sage.rings.asymptotic.term_monoid.GenericTermMonoid method*), 100
cls (*sage.rings.asymptotic.growth_group.GrowthGroupFactor attribute*), 66
coefficient_ring (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing attribute*), 33
coefficient_ring (*sage.rings.asymptotic.term_monoid.GenericTermMonoid attribute*), 101
coefficients_of_generating_function() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 33
coerce_point() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_functions*), 149
cohomology_decomposition() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 133
combine_exceptions() (*in module sage.rings.asymptotic.misc*), 115
compare_with_values() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 16
construction() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 34
construction() (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup method*), 54
construction() (*sage.rings.asymptotic.growth_group.ExponentialNonGrowthGroup method*), 56
construction() (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup method*), 68
construction() (*sage.rings.asymptotic.growth_group.MonomialNonGrowthGroup method*), 71
construction() (*sage.rings.asymptotic.term_monoid.BTerm method*), 86
construction() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 96
construction() (*sage.rings.asymptotic.term_monoid.TermWithCoefficient method*), 109
create_key_and_extra_args() (*sage.rings.asymptotic.growth_group.GrowthGroupFactory method*), 67
create_key_and_extra_args() (*sage.rings.asymptotic.growth_group_cartesian.CartesianProductFactory method*), 75
create_key_and_extra_args() (*sage.rings.asymptotic.term_monoid.TermMonoidFactory method*), 108
create_object() (*sage.rings.asymptotic.growth_group.GrowthGroupFactory method*), 67
create_object() (*sage.rings.asymptotic.growth_group_cartesian.CartesianProductFactory method*), 75
create_object() (*sage.rings.asymptotic.term_monoid.TermMonoidFactory method*), 108
create_summand() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 37
critical_cone() (*sage.rings.asymptotic.asymptotics_multivariate_genera method*), 133

D

DecreasingGrowthElementError, 52
default_locals() (*sage.rings.asymptotic.misc.Locals method*), 113
default_prec (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing attribute*), 35
DefaultTermMonoidFactory (*in module sage.rings.asymptotic.term_monoid*), 88
denominator() (*sage.rings.asymptotic.asymptotics_multivariate_generati method*), 134
denominator_factored() (*sage.rings.asymptotic.asymptotics_multivariate_generating_func method*), 134
denominator_ring (*sage.rings.asymptotic.asymptotics_multivariate_gene attribute*), 135
denominator_ring (*sage.rings.asymptotic.asymptotics_multivariate_gene attribute*), 148
diff_all() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 150
diff_op() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 151
diff_op_simple() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 152
diff_prod() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 153
diff_seq() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 154
dimension() (*sage.rings.asymptotic.asymptotics_multivariate_generating_ method*), 135
direction() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 154
DivisionRings (*sage.rings.asymptotic.growth_group.ExponentialGrowthG attribute*), 54

E

Element (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing attribute*), 32
Element (*sage.rings.asymptotic.asymptotics_multivariate_generating_funct attribute*), 147
Element (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup attribute*), 54
Element (*sage.rings.asymptotic.growth_group.ExponentialNonGrowthGrou attribute*), 56

Element (*sage.rings.asymptotic.growth_group.GenericGrowthGroup attribute*), 61
Element (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup attribute*), 68
Element (*sage.rings.asymptotic.growth_group.MonomialNonGrowthGroup attribute*), 71
Element (*sage.rings.asymptotic.term_monoid.BTermMonoid attribute*), 87
Element (*sage.rings.asymptotic.term_monoid.ExactTermMonoid attribute*), 94
Element (*sage.rings.asymptotic.term_monoid.GenericTermMonoid attribute*), 100
Element (*sage.rings.asymptotic.term_monoid.OTermMonoid attribute*), 107
Element (*sage.rings.asymptotic.term_monoid.TermWithCoefficientMonoid attribute*), 110
error_part() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 17
exact_part() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 17
ExactTerm (*class in sage.rings.asymptotic.term_monoid*), 88
ExactTermMonoid (*class in sage.rings.asymptotic.term_monoid*), 93
exp() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 17
exp() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct.Element method*), 76
exponent (*sage.rings.asymptotic.growth_group.MonomialGrowthElement attribute*), 68
ExponentialGrowthElement (*class in sage.rings.asymptotic.growth_group*), 52
ExponentialGrowthGroup (*class in sage.rings.asymptotic.growth_group*), 53
ExponentialGrowthGroupFunctor (*class in sage.rings.asymptotic.growth_group*), 56
ExponentialNonGrowthElement (*class in sage.rings.asymptotic.growth_group*), 56
ExponentialNonGrowthGroup (*class in sage.rings.asymptotic.growth_group*), 56
ExponentialNonGrowthGroupFunctor (*class in sage.rings.asymptotic.growth_group*), 57
extend_by_non_growth_group (*sage.rings.asymptotic.growth_group.GrowthGroupFactor attribute*), 66
extended_by_non_growth_group() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 61
extract_variable_names() (*sage.rings.asymptotic.growth_group.Variable static method*), 72

F

factorial() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 18
factors() (*sage.rings.asymptotic.growth_group.GenericGrowthElement method*), 57
factors() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 76
factory() (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup class method*), 54
factory() (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup class method*), 69
FractionWithFactoredDenominator (*class in sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 123
FractionWithFactoredDenominatorRing (*class in sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 146
FractionWithFactoredDenominatorSum (*class in sage.rings.asymptotic.asymptotics_multivariate_generating_funct*), 147
from_construction() (*sage.rings.asymptotic.term_monoid.GenericTermMonoid method*), 101

G

gen() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 35
gen() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 62
GenericGrowthElement (*class in sage.rings.asymptotic.growth_group*), 57
GenericGrowthGroup (*class in sage.rings.asymptotic.growth_group*), 61
GenericNonGrowthElement (*class in sage.rings.asymptotic.growth_group*), 65
GenericNonGrowthGroup (*class in sage.rings.asymptotic.growth_group*), 65
GenericProduct (*class in sage.rings.asymptotic.growth_group_cartesian*), 75
GenericProduct.Element (*class in sage.rings.asymptotic.growth_group_cartesian*), 76
GenericTerm (*class in sage.rings.asymptotic.term_monoid*), 94
GenericTermMonoid (*class in sage.rings.asymptotic.term_monoid*), 99
gens() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 36
gens() (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup method*), 55
gens() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 62
gens_logarithmic() (*sage.rings.asymptotic.growth_group.MonomialGro method*), 69

gens_monomial() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 63
gens_monomial() (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup method*), 69
gens_monomial() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 80
grads() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions method*), 136
Groups (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup attribute*), 54
growth_group (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing attribute*), 36
growth_group (*sage.rings.asymptotic.term_monoid.GenericTermMonoid attribute*), 102
GrowthGroup (*in module sage.rings.asymptotic.growth_group*), 65
GrowthGroupFactor (*class in sage.rings.asymptotic.growth_group*), 66
GrowthGroupFactory (*class in sage.rings.asymptotic.growth_group*), 66

H

HarmonicNumber() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 41
has_same_summands() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 19

I

ImplicitExpansion() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 41
ImplicitExpansionPeriodicPart() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 43
InverseFunctionAnalysis() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 44
invert() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 19
is_compatible() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 63
is_compatible() (*sage.rings.asymptotic.growth_group.PartialConversionElement method*), 71
is_constant() (*sage.rings.asymptotic.term_monoid.ExactTerm method*), 90
is_constant() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 96
is_convenient_multiple_point() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 136
is_exact() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 20
is_exact() (*sage.rings.asymptotic.term_monoid.ExactTerm method*), 90
is_exact() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 97
is_Generic() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 20
is_little_o_of_one() (*sage.rings.asymptotic.term_monoid.ExactTerm method*), 91
is_little_o_of_one() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 97
is_little_o_of_one() (*sage.rings.asymptotic.term_monoid.OTerm method*), 104
is_lt_one() (*sage.rings.asymptotic.growth_group.GenericGrowthElement method*), 57
is_lt_one() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 77
is_monomial() (*sage.rings.asymptotic.growth_group.Variable method*), 73

L

le() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 63
le() (*sage.rings.asymptotic.term_monoid.GenericTermMonoid method*), 102
leinartas_decomposition() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 137
limit() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 21
locals (*class in sage.rings.asymptotic.misc*), 113
locals() (*sage.rings.asymptotic.misc.WithLocals method*), 114
log() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 21
log() (*sage.rings.asymptotic.growth_group.GenericGrowthElement method*), 58
log() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct.Element method*), 77
log_factor() (*sage.rings.asymptotic.growth_group.GenericGrowthElement method*), 59
log_factor() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 78
log_grads() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 138
log_Stirling() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 48
log_string() (*in module sage.rings.asymptotic.misc*), 115
log_term() (*sage.rings.asymptotic.term_monoid.ExactTerm method*), 91

log_term() (*sage.rings.asymptotic.term_monoid.GenericTermMonoid method*), 98
log_term() (*sage.rings.asymptotic.term_monoid.OTermMonoid method*), 105

M

maclaurin_coefficients() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator method*), 139
Magmas (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup attribute*), 54
Magmas (*sage.rings.asymptotic.growth_group.GenericGrowthGroup attribute*), 61
Magmas (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup attribute*), 68
map_coefficients() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 23
merge() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRingFunctor method*), 38
merge() (*sage.rings.asymptotic.growth_group.AbstractGrowthGroupFunctor method*), 52
module
    sage.rings.asymptotic.asymptotic_expansion_generators, 39
    sage.rings.asymptotic.asymptotic_ring, 7
    sage.rings.asymptotic.asymptotics_multivariate_generating_functions, 120
    sage.rings.asymptotic.growth_group, 48
    sage.rings.asymptotic.growth_group_cartesian, 74
    sage.rings.asymptotic.misc, 112
    sage.rings.asymptotic.term_monoid, 82
monomial_coefficient() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 23
MonomialGrowthElement (*class in sage.rings.asymptotic.growth_group*), 67
MonomialGrowthGroup (*class in sage.rings.asymptotic.growth_group*), 68
MonomialGrowthGroupFunctor (*class in sage.rings.asymptotic.growth_group*), 70
MonomialNonGrowthElement (*class in sage.rings.asymptotic.growth_group*), 70
MonomialNonGrowthGroup (*class in sage.rings.asymptotic.growth_group*), 70
MonomialNonGrowthGroupFunctor (*class in sage.rings.asymptotic.growth_group*), 71
MultivariateProduct (*class in sage.rings.asymptotic.growth_group_cartesian*), 81

N

ngens() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 36
ngens() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 64
NoConvergenceError, 39
non_growth_group() (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup method*), 55
non_growth_group() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 64
non_growth_group() (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup method*), 70
NotImplementedBZero, 113
NotImplementedOZero, 113
nullstellensatz_certificate() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator method*), 140
nullstellensatz_decomposition() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator method*), 140
numerator() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator method*), 141
numerator_ring (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator attribute*), 142

O

O() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 15
OTerm (*in module sage.rings.asymptotic.term_monoid*), 103
OTermMonoid (*class in sage.rings.asymptotic.term_monoid*), 107

P

parent_to_repr_short() (*in module sage.rings.asymptotic.misc*), 116
PartialConversionElement (*class in sage.rings.asymptotic.growth_group*), 71
PartialConversionValueError, 71
permutation_sign() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator*), 154
plot_comparison() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 23
Posets (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup attribute*), 54
Posets (*sage.rings.asymptotic.growth_group.GenericGrowthGroup attribute*), 61
Posets (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup attribute*), 68
pow() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 24

Q

quotient() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactorialDenominator method*), 142

R

relative_error() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 143
repr_op() (*in module sage.rings.asymptotic.misc*), 116
repr_short_to_parent() (*in module sage.rings.asymptotic.misc*), 117
rpow() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 25
rpow() (*sage.rings.asymptotic.growth_group.GenericGrowthElement method*), 59
rpow() (*sage.rings.asymptotic.growth_group_cartesian.GenericProductElement method*), 79
rpow() (*sage.rings.asymptotic.term_monoid.ExactTerm method*), 92
rpow() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 99
rpow() (*sage.rings.asymptotic.term_monoid.OTerm method*), 106

S

sage.rings.asymptotic.asymptotic_expansion_generators module, 39
sage.rings.asymptotic.asymptotic_ring module, 7
sage.rings.asymptotic.asymptotics_multivariate_generating_functions module, 120
sage.rings.asymptotic.growth_group module, 48
sage.rings.asymptotic.growth_group_cartesian module, 74
sage.rings.asymptotic.misc module, 112
sage.rings.asymptotic.term_monoid module, 82
Sets (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup attribute*), 54
Sets (*sage.rings.asymptotic.growth_group.GenericGrowthGroup attribute*), 61
Sets (*sage.rings.asymptotic.growth_group.MonomialGrowthGroup attribute*), 68
show() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 26
singular_ideal() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 143
SingularityAnalysis() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 45
smooth_critical_ideal() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 144
some_elements() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 37
some_elements() (*sage.rings.asymptotic.growth_group.ExponentialGrowthGroup method*), 55
some_elements() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 64
some_elements() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 81
some_elements() (*sage.rings.asymptotic.term_monoid.BTermMonoid method*), 88
some_elements() (*sage.rings.asymptotic.term_monoid.GenericTermMonoid method*), 102
some_elements() (*sage.rings.asymptotic.term_monoid.TermWithCoefficient method*), 110
split() (*sage.rings.asymptotic.growth_group.PartialConversionElement method*), 71
split_str_by_op() (*in module sage.rings.asymptotic.misc*), 117
sqrt() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 26
Stirling() (*sage.rings.asymptotic.asymptotic_expansion_generators.AsymptoticExpansionGenerators static method*), 47
strip_symbolic() (*in module sage.rings.asymptotic.misc*), 117
subs() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 26
subs_all() (*in module sage.rings.asymptotic.asymptotics_multivariate_generating_functions*), 155
substitute() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 27
substitute_raise_exception() (*in module sage.rings.asymptotic.misc*), 118
sum() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 148
summands (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion attribute*), 28
symbolic_expression() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 29

T

term_monoid() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 37
term_monoid() (*sage.rings.asymptotic.term_monoid.GenericTermMonoid method*), 102
term_monoid_factory (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing attribute*), 37
term_monoid_factory (*sage.rings.asymptotic.term_monoid.GenericTermMonoid attribute*), 103
TermMonoidFactory (*class in sage.rings.asymptotic.term_monoid*), 107
TermWithCoefficient (*class in sage.rings.asymptotic.term_monoid*), 109
TermWithCoefficientMonoid (*class in sage.rings.asymptotic.term_monoid*), 110

transform_category() (*in module sage.rings.asymptotic.misc*), 118
truncate() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 29

U

univariate_decomposition() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominator method*), 145
UnivariateProduct (*class in sage.rings.asymptotic.growth_group_cartesian*), 82

V

var (*sage.rings.asymptotic.growth_group.GrowthGroupFactor attribute*), 66
Variable (*class in sage.rings.asymptotic.growth_group*), 72
variable_names() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticExpansion method*), 30
variable_names() (*sage.rings.asymptotic.asymptotic_ring.AsymptoticRing method*), 38
variable_names() (*sage.rings.asymptotic.growth_group.GenericGrowthElement method*), 60
variable_names() (*sage.rings.asymptotic.growth_group.GenericGrowthGroup method*), 65
variable_names() (*sage.rings.asymptotic.growth_group.Variable method*), 73
variable_names() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct method*), 81
variable_names() (*sage.rings.asymptotic.growth_group_cartesian.GenericProduct.Element method*), 80
variable_names() (*sage.rings.asymptotic.term_monoid.GenericTerm method*), 99

W

whole_and_parts() (*sage.rings.asymptotic.asymptotics_multivariate_generating_functions.FractionWithFactoredDenominatorSum method*), 149
WithLocals (*class in sage.rings.asymptotic.misc*), 113

Z

ZeroCoefficientError, 111